I. Moerdijk: Dendroidal sets and test categories

Two new examples of test categories (with Cisinski)

A few definitions from Astérisque 301.

Let A be a small category:

$$\begin{tikzcd} Sets^{A^{op}} \arrow[r, shift right, "i_{A!}"'] \arrow[dr, "\lambda_{A!}"] & Cat \arrow[l, shift right, "i_A^*"'] \arrow[d, "N \leftarrow \text{nerve}"] \\ & sSets \arrow[ul, bend left, "\lambda_A^*"] \end{tikzcd}$$

$\leftarrow$ category of elements functor with his right adjoint $i_{A!} \dashv i_A^*$

$\dashv \lambda_A^* \rightarrow$

…et: A is called test category if:

① A is contractible (i.e. $\|NA\|$ is)

② For any object $a \in A$ & the corresponding slice A/a, $\forall \mathbb{C}$ small cat:

$$i_{A/a!}\, i^*_{A/a}(\mathbb{C}) \xrightarrow{\text{counit}} \mathbb{C}$$

is a weak equivalence.

…uillen's theorem A gives that it is enough to check that each $\mathbb{C}/x$ has this property; equally: $i_{(A/a)!}\, i^*_{(A/a)}(\mathbb{C}/x)$ is contractible.

It also follows that:

$$i_{A!}\, i_{A^*}(\mathbb{C}) \longrightarrow \mathbb{C} \text{ is a weak equivalence}$$

(or $i_{A!}\, i_A^*(\mathbb{C}/x)$ contractible)

Remark: If $\mathbb{C} = \{c\}$ we find that $i_{A!}\, i_A^*(\{c\}) = A$.

For $\mathbb{C} = \{c \to 1\}$ we find that

$i_A^*\{c \to 1\} = L$, the subobject classifier, or "Lawvere object" of $Sets^{A^{op}}$

$\Rightarrow$ We find by considering L-parametrized homotopies that A is test iff it is contractible and each $i_{A!}(L/a)$ is, where

$$L/a = L \times A(-,a), = \text{subobject classifier of } Sets^{(A/a)^{op}}.$$

Problem: sSets are good for homotopy theory. What categories A bring "good homotopy theory" in $Sets^{A^{op}}$ (sSets = case $A = \Delta$).

Theorem: if A is test, then there is a Quillen model structure on $Sets^{A^{op}}$, for which:

$\rightarrow$ cofibrations $\equiv$ mono morphisms

$\rightarrow$ the pair $\lambda_{A!} : Sets^{A^{op}} \rightleftarrows sSets : \lambda_A^*$ is a Quillen equivalence.

Remark: the converse is also true!

Examples,

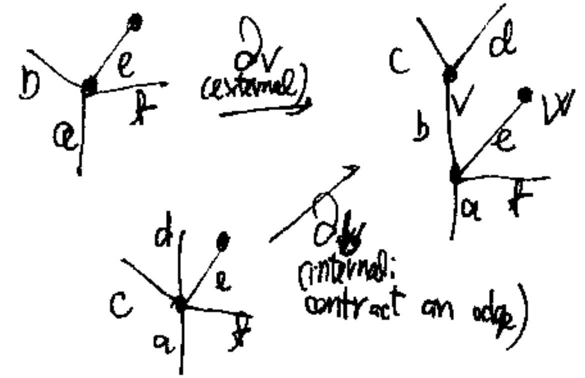
• About dendroidal sets. For A, we take "the" category of trees, Ω:

Objects = finite rooted trees, with leaves.

Arrows: come in 3 kinds

(a) Isomorphisms

(b) Face maps – external
– internal



(c) degeneracies (....)

Every such tree T generates a coloured operad $\Omega(T)$, and the morphisms in Ω are precisely the morphisms of operads.

$Sets^{\Omega^{op}}$ = dSets = dendroidal sets. This is an extension of $Sets^{\Delta^{op}}$ = sSets = simplicial sets, via the inclusion

$i: \Delta \to \Omega \quad [n] \longmapsto$ 

Aim: combinatorial model for topological operads.

Theorem (w/ Cisinski):

(a) The category dSets has a Quillen model structure, cofibrations = monomorphisms

(b) There is a Quillen equivalence

$$dSets \rightleftarrows \text{simplicial operads (or topological operads)}$$

(c) If you slice (a) and (b) over $i(\Delta(c)) = \eta$, then (a) restricts to the Joyal model structure on sSets, and (b) restricts to a Quillen equivalence

$$sSets \rightleftarrows \text{simpl.cat. (or top'l categories)}$$

Theorem: Ω is a test category.

Prop. 1: Ω is contractible.

Proof: Let $\bar{\Omega}$ = category of closed trees (i.e. trees w/o leaves). The inclusion $\bar{\Omega} \hookrightarrow \Omega$ has a left adjoint $cl: \Omega \to \bar{\Omega}$ (close the leaves).

$$\begin{tikzcd} \bar{\Omega} \arrow[r, bend left, "id"] \arrow[r, "V"] \arrow[r, bend right] & \bar{\Omega} \arrow[r, hook, "incl"] & \Omega \arrow[l, bend right, "cl"'] \end{tikzcd}$$

$\alpha \Downarrow \quad \beta \Uparrow$

We look to $\bar{\Omega}$. In $\bar{\Omega}$ we have $V(T) = T$ w/ new vertex at the root.

$T \xrightarrow{\alpha} V(T) \xleftarrow{\beta} \; |_r = \bar{\eta} = cl(\eta)$

These are natural in $\bar{\Omega} \Rightarrow \bar{\Omega}$ is contractible.

Corollary (by a criterion of C+M): $\bar{\Omega}$ is a test category.

~ Proof that Ω is a test category. It is enough to prove that Ω/T is test.

Idea. $sSets \underset{\sim}{\xrightarrow{con}} (sSets^{(\Omega/T)^{op}})_{proj,con} \xrightarrow{\sim} (sSets^{(\Omega/T)^{op}})_{Reedy,con} \xrightarrow{\lambda_!} sSets/\lambda_! T$

Quillen eq

$\Rightarrow \lambda_!$ is a Quillen

$dSets/T_{con} \xrightarrow{\sim} (sSets^{(\Omega/T)^{op}})_{Reedy, con, constr. Segal}$

$dSets/T \to sSets^{(\Omega/T)^{op}}$