Vertical Categorification of classical AQFT

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- (Einstein) **Causality**: If U, V are spacelike separated, then $\mathcal{A}(U)$ and $\mathcal{A}(V)$ pairwise commute in the *quasilocal algebra*

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We seek a **categorification** of this notion.

Categorification

Vertical categorification serves (in the words of John Baez)

> *as a tool to find category-theoretic analogs of set-theoretic concepts by replacing* **sets** *with* **categories**, **functions** *with* **functors**, *and* **equations** *between functions by* **natural isomorphisms** *between functors, which in turn should satisfy certain equations of their own, called "coherence laws".*

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Locality is encoded by assigning to any open subset of M an entire C^*-category. How can we categorify Einstein causality? [Comeau] proposes a theory in such a way that causality axiom corresponds to an higher-categorical analogue of the notion of *Von Neumann algebra*, a subalgebra $A \leq B(H)$ which equals its *double commutant* A''.

Recall that

Definition

A (strict) *monoidal category* consists of a category **C** with a tensor $\otimes$, in which we can find a distinguished object $\mathbb{I}$, to be called *unit object* such that

$$V \otimes \mathbb{I} = V = \mathbb{I} \otimes V \qquad (\forall V \in \mathrm{Ob}_{\mathbf{C}})$$

and such that for any three U, V, W one has $U \otimes (V \otimes W) = (U \otimes V) \otimes W$.

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Strict monoidal categories are rather rare structures, but weaken the axiom asking that $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ is not a “real” generalization:

Theorem (Mac Lane's coherence)

Any monoidal category is equivalent to a strict one.

Monoidal functors $\mathcal{F} : (\mathbf{C}, \otimes_{\mathbf{C}}) \to (\mathbf{D}, \otimes_{\mathbf{D}})$ respect this structure:

$$\begin{array}{ccc}
(\mathcal{F}(A) \otimes_{\mathbf{D}} \mathcal{F}(B)) \otimes_{\mathbf{D}} \mathcal{F}(C) & \longrightarrow & \mathcal{F}(A) \otimes_{\mathbf{D}} (\mathcal{F}(B) \otimes_{\mathbf{D}} \mathcal{F}(C)) \\
\downarrow & & \downarrow \\
\mathcal{F}(A \otimes_{\mathbf{C}} B) \otimes_{\mathbf{D}} \mathcal{F}(C) & & \mathcal{F}(A) \otimes_{\mathbf{D}} \mathcal{F}(B \otimes_{\mathbf{C}} C) \\
\downarrow & & \downarrow \\
\mathcal{F}((A \otimes_{\mathbf{C}} B) \otimes_{\mathbf{C}} C) & \longrightarrow & \mathcal{F}(A \otimes_{\mathbf{C}} (B \otimes_{\mathbf{C}} C))
\end{array}$$

PMCs as #-monoids

(Well known) fact: the (2-)category **Cat** is a complete and cocomplete closed symmetric monoidal category, with respect to the "cartesian product of categories" tensor and where the internal-hom is given exactly by the category of functors between two fixed categories.

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(less notorious) fact: **Cat** admits *exactly one* different cosmos structure, where the tensor is given by the category $\mathbf{C}\#\mathbf{D}$ having the same objects as $\mathbf{C} \times \mathbf{D}$ and where the set of morphisms between (C, D) and (C', D') is given by the set of "directed paths" with a suitable composition law, in such a way that

> $\mathbf{C}\#\mathbf{D}$ *is the unique category* $\mathbf{X}$ *equipped with two families of functors* $\{\mathcal{F}_C \colon \mathbf{D} \to \mathbf{X}\}_{C \in \mathrm{Ob}_\mathbf{C}}, \{\mathcal{G}_D \colon \mathbf{C} \to \mathbf{X}\}_{D \in \mathrm{Ob}_\mathbf{D}}$ *such that* $\mathcal{F}_C(D) = \mathcal{G}_D(C)$ *for any* $(C, D) \in \mathrm{Ob}_{\mathbf{C}\times\mathbf{D}}$.

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Crude definition: A binoidal category as an *internal magma* in $\mathbf{Cat}_\#$ (=the (2-)category **Cat** endowed with the #-symmetric monoidal structure); a (strict) premonoidal category is a monoid in $\mathbf{Cat}_\#$ (in the same way a monoidal category was a monoid in $\mathbf{Cat} = \mathbf{Cat}_\times$).

PMCs as #-monoids

Unraveling the crude definition:

Definition

A *binoidal category* is a category $\mathbf{C}$ with two families of $\mathrm{Ob}_{\mathbf{C}}$-indexed endofunctors $\{R_A, L_A\}_{A \in \mathrm{Ob}_{\mathbf{C}}}$, such that $R_B(A) = L_A(B)$ for any $A, B \in \mathrm{Ob}_{\mathbf{C}}$.

The object $R_B(A) = L_A(B)$ is often denoted $A \otimes B$, R_B as $- \otimes B$ and L_A as $A \otimes -$.

The correspondence $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is called *pretensor*: it is said to be *associative* if there exists a (natural) isomorphim $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ for any three objects A, B, C. Finally we denote $L_A(f) = 1_A \otimes f = A \otimes f$ and $R_B(f) = f \otimes 1_B = f \otimes B$.

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> **Intuition**: *a premonoidal category (PMC for short) is a monoidal category where the pretensor $\otimes$ is not bifunctorial, albeit being functorial when saturated in each of its two "arguments".*

In a binoidal category there are *two* different ways to compose a pair of arrows: denoting the product "$f \otimes g$" is ambiguous. Absence of bifunctoriality is the key point to categorify the notion of commutant.

Definition (Right and left product)

Suppose $(\mathbf{C}, \otimes)$ is binoidal, and define for any $f\colon A \to C, g\colon B \to D$ the **right** and **left product** of f and g, g and f, to be

$$g \rtimes f := (g \otimes 1_C) \circ (1_B \otimes f) \qquad g \ltimes f := (1_D \otimes f) \circ (g \otimes 1_A).$$

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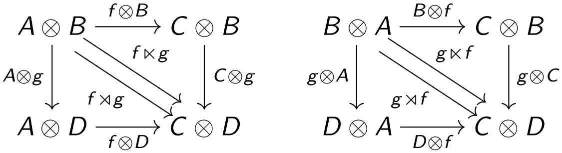
Definition (Central Morphisms)

Suppose $(\mathbf{C}, \otimes)$ is binoidal, we say that $f\colon A \to C$ is *central* if for any $g\colon B \to D$ one has $g \rtimes f = g \ltimes f$ and $f \rtimes g = f \ltimes g$.
A natural transformation $\alpha\colon \mathcal{G} \Rightarrow \mathcal{H}$ between functors $\mathcal{G}, \mathcal{H}\colon (\mathbf{B}, \otimes_{\mathbf{B}}) \to (\mathbf{C}, \otimes_{\mathbf{C}})$ is said to be central if every α_A is a central map.



Definition

A PMC consists of a binoidal category $(\mathbf{C}, \otimes_{\mathbf{C}})$, where $\otimes = \otimes_{\mathbf{C}}$ is an associative pretensor, and an object $I \in \mathrm{Ob}_{\mathbf{C}}$ with central natural equivalences $\alpha\colon (- \otimes -) \otimes - \Longrightarrow - \otimes (- \otimes -)$, $\lambda\colon - \otimes I \Longrightarrow \mathrm{id}_{\mathbf{C}}$, $\varrho\colon I \otimes - \Longrightarrow \mathrm{id}_{\mathbf{C}}$, satisfying the exact formal analogue of the coherence conditions for a monoidal category.

Definition

Let $(\mathbf{C}, \otimes)$ be a PMC. Its *center* $Z(\mathbf{C})$ consists of the subcategory with the same objects, where $\hom_{Z(\mathbf{C})}(A, B)$ is made by all central arrows between A and B.

center of a monoid $(M, \cdot)$ $\overset{\text{categorification}}{\rightsquigarrow}$ center of a PMC $\mathbf{C}$:

Any monoid $(M, \cdot)$ can be regarded as a premonoidal category with a single object $*$ and such that $\mathrm{End}(*) = M$; the tensor product is the operation of M, and this category is monoidal iff this operation is *commutative*.

The category $\mathbf{Hilb}_{\otimes H}$

Example

Let $(\mathbf{C}, \otimes, I)$ a symmetric **monoidal** category with simmetry τ, let $S \in \mathrm{Ob}_{\mathbf{C}}$. Define a new category $\mathbf{C}_{\otimes S}$ with the same objects of $\mathbf{C}$, and where $\hom_{\mathbf{C}_{\otimes S}}(X, Y) = \hom_{\mathbf{C}}(X \otimes S, Y \otimes S)$. Define $Z \otimes f = 1_Z \otimes f$ and $f \otimes Z$ to be

$$X \otimes Z \otimes S \xrightarrow{\tau_{XZ} \otimes 1_S} Z \otimes X \otimes S \xrightarrow{1_Z \otimes f} Z \otimes Y \otimes S \xrightarrow{\tau_{ZY} \otimes 1_S} Y \otimes Z \otimes S.$$

With these definitions, $\mathbf{C}_{\otimes S}$ is premonoidal in view of the monoidal structure on $\mathbf{C}$.

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Application to $(\mathbf{C}, \otimes, I, S) = (\mathbf{Hilb}, \otimes_{\mathbb{C}}, \mathbb{C}, H)$: quantum processes amount to maps $X \to Y$ in **Hilb**; in the premonoidal setting H plays the rôle of the environment, and maps $X \otimes H \to Y \otimes H$ take into account interaction with the environment.
$f \in \mathbf{Hilb}_{\otimes H}(X, Y)$ is central $\iff$ no interaction.

Proposition

Every central morphism $f \in \hom_{\mathbf{Hilb}_{\otimes H}}(X, Y)$ comes from $\widehat{f} \in \hom_{\mathbf{Hilb}}(X, Y)$, via $f = \hat{f} \otimes 1_H$.

Von Neumann categories

Definition

A *positive* **-operation* on a $\mathbb{C}$-linear PMC $\mathbf{C}$ consists of an antiequivalence $(-)^*\colon \mathbf{C}^{\text{op}} \to \mathbf{C}$ such that

- It is the identity on objects and an antilinear map on the level of morphisms;
- $(-)^{**} = \text{id}_{\mathbf{C}}$;
- for any $f\colon X \to Y$, the two arrows $f^* \circ f\colon X \to X$ and $f \circ f^*\colon Y \to Y$ are the zero map iff $f = 0_{XY}$.

A *premonoidal* **-category* consists of a $\mathbb{C}$-linear premonoidal category endowed with a positive *-operation. Finally a *premonoidal* C^**-category* $(\mathbf{C}, \otimes, I, \| - \|_\bullet)$ consists of a premonoidal Banach-*-algebra-enriched $\mathbb{C}$-linear category, such that

$$\|g \circ f\|_{XZ} \le \|g\|_{YZ} \cdot \|f\|_{XY}, \quad \|f^* \circ f\|_X = \|f \circ f^*\|_Y = \|f\|^2_{XY}$$

for any two morphisms of $\mathbf{C}$ $f\colon X \to Y$, $g\colon Y \to Z$.

Von Neumann categories

Definition

Let $\mathbf{A} \subseteq \mathbf{C}$ be a subcategory of a PM *-category $\mathbf{C}$; the *commutant* of $\mathbf{A}$, denoted $\mathbf{A}'$, is the subcategory of $\mathbf{C}$ with the same class of objects and
$\mathbf{A}'(X, Y) = \{f\colon X \to Y \mid f \ltimes g = f \rtimes g,\ g \ltimes f = g \rtimes f \text{ for all } g \in \mathrm{Mor}(\mathbf{A})\}$.

commutant of $A \leq B(H)$ $\overset{\text{categorification}}{\rightsquigarrow}$ commutant of $\mathbf{A} \subseteq \mathbf{C}$

Theorem

The commutant $\mathbf{A}'$ of $\mathbf{A} \subseteq \mathbf{C}$ is a *-premonoidal category provided $\mathbf{A}$ is such a category.

Sketch of proof.

Composition of arrows in $\mathbf{A}'$ remains in $\mathbf{A}'$, and every coherence condition (associativity and unit diagrams) holds because it involves arrows living in the center $Z(\mathbf{C}) \subseteq \mathbf{A}'$. □

We got it!

Definition

Let $\mathbf{A} \subseteq \mathbf{C}$ be a premonoidal C^*-subcategory of a premonoidal C^*-category $\mathbf{C}$; then $\mathbf{A}$ is called a $\mathbf{C}$-*Von Neumann category* if $\mathbf{A}''(X, Y) = \mathbf{A}(X, Y)$ for any $X, Y \in \mathrm{Ob}_{\mathbf{A}}$. When the context is clear, or when $\mathbf{C} = \mathbf{Hilb}_{\otimes H}$, a $\mathbf{C}$-Von Neumann category is simply said *Von Neumann*.

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Proposition

If $\mathbf{A}$ is a Von Neumann category then $\mathbf{A}(\mathbb{C}, \mathbb{C})$ is a Von Neumann algebra ($\Longrightarrow$ every one-object Von Neumann category is a Von Neumann algebra).

Proposition

If $\mathbf{A}$ is a $(-)^*$-closed subcategory of a premonoidal C^*-category, then $\mathbf{A}'$ itself is a premonoidal C^*-category, and in particulat a $\mathbf{C}$-Von Neumann category.

Algebraic approach to QFT: BF

Definition (Category of regions)

Loc has objects all smooth d-dimensional, globally hyperbolic, lorentzian, oriented and time-oriented manifolds (M, g); for any two such M, N, the set $\mathbf{Loc}(M, N)$ consists of all the isometric embeddings $\iota \colon M \to N$ subject to the following constraints:

- The subspace $M \cong \iota M \subseteq N$ is *causally convex*, i.e. it contains every causal curve whose endpoints are in ιM.
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Definition (Category of observables)

The category **Obs** of observables is formed by all unital complex C^*-algebras, and $\mathbf{Obs}(C, D)$ are the injective unital *-morphisms $C \to D$; the composition amounts to composition of maps.

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Definition ((Locally covariant) QFT)

A *locally covariant Quantum Field Theory* is an object of the (2-)category $\mathcal{A} \in \mathrm{Funct}(\mathbf{Loc}, \mathbf{Obs})$, often denoted $[\mathbf{Loc}, \mathbf{Obs}]$ for short.

Algebraic approach to QFT: AHK

A slightly less general path than that in [Brunetti-Fredenhagen] lead to the **AHK-axioms** for AQFTs: fix once and for all an object $M \in \mathbf{Loc}$ and consider the category $\mathbf{K}(M)$ of all open, relatively compact and causally convex subsets of M. This becomes a poset in the obvious way.

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Proposition (Araki-Haag-Kastler AQFTs)

Let $\mathcal{A}$ a locally covariant QFT; define $\tilde{\mathcal{A}} \colon \mathbf{K}(M) \to \mathbf{Obs}$ sending $U \mapsto \tilde{\mathcal{A}}(U) = \mathcal{A}(U) \subseteq \mathcal{A}(M)$; then

- $U \subseteq V$ implies $\tilde{\mathcal{A}}(U) \subseteq \tilde{\mathcal{A}}(V)$ for any two $U, V \in \mathbf{K}(M)$;
- $[\tilde{\mathcal{A}}(U), \tilde{\mathcal{A}}(V)] = 0$ for any two causally separated $U, V \in \mathbf{K}(M)$.

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$\tilde{\mathcal{A}}\colon \mathbf{K}(M) \to \mathbf{Obs}$ defines a *copresheaf* on $\mathbf{K}(M)$. The essential image of this functor is contained in the posetal category $\mathrm{C}^*_{\subseteq}(M) \le \mathbf{Obs}$, whose objects are the C^*-subalgebras of $\mathcal{A}(M)$ and whose morphisms are unital embeddings, so any AHK-QFT can be regarded as a $\mathrm{C}^*_{\subseteq}(M)$-valued presheaf on $\mathbf{K}(M)$.

Definition

Let $(K, \preceq)$ be a directed poset; then a *local system of premonoidal* C^**-categories* consists of a functor $\mathcal{A}\colon K \to \mathrm{pC}^*$-**Cat** from K to the posetal 2-category pC^*-**Cat**$_\subseteq$ of premonoidal C^*-categories and premonoidal *faithful* C^*-functors between them.

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Define the categorified quasilocal algebra to be

$$\mathcal{A}^\circ := \Big(\bigsqcup_{U\in K} \mathcal{A}(U)\Big)/\!\sim$$

where $(U, A \in \mathrm{Ob}_{\mathcal{A}(U)}) \sim (V, B \in \mathrm{Ob}_{\mathcal{A}(V)})$ iff there exists a $W \succcurlyeq U, V$ such that the objects $i_{UW}(A)$ and $i_{VW}(B)$ are equal in $\mathcal{A}(W)$, and similarly identifies $(U, f), (V, g)$ if they are "eventually" equal.

Lemma

If $\mathcal{A}$ is a local system of premonoidal C^*-categories, then $\mathcal{A}^\circ$ is a PMC.

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Lemma

If $\mathcal{A}$ is a local system of premonoidal C^*-categories, then $\mathcal{A}^\circ$ is a PMC.

$\mathcal{A}^\circ \not\cong \varinjlim_{U \in K} \mathcal{A}(U)$! (but it's not so far...)

$\mathcal{A}^\circ$ embeds into a PM C^*-category $\mathbf{U}(\mathcal{A})$ with a faithful-bijective-on-objects functor; the clarifying example is a local system $\mathcal{A}$ of C^*-algebras (=one-objects PM C^*-categories):

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We now want to insert Einstein causality: let $(K, \leq)$ be the poset of open double cones in Minkowski space, ordered by inclusion of subsets.

Definition (Premonoidal C^*-QFT)

A *premonoidal* C^*-QFT consists of a local system of premonoidal C^*-categories $\mathcal{A} \colon (K, \leq) \to \mathrm{pC}^*\text{-}\mathbf{Cat}$ subject to the additional condition of Einstein's causality:

Whenever $U \perp V$, then $\mathcal{A}(U)$ and $\mathcal{A}(V)$ commute in $\mathbf{U}(\mathcal{A})$, namely $\mathcal{A}(U)' \geq \mathcal{A}(V)$, $\mathcal{A}(V)' \geq \mathcal{A}(U)$.

Suppose $M \leq B(H)$ is a G-Von Neumann algebra for some discrete group G.

Definition (Crossed Product of VNA)

The crossed product Von Neumann algebra $M \curlyvee G = \widetilde{M}_G$ is defined via two embeddings

$$M \xrightarrow{\pi} M \curlyvee G \xleftarrow{\lambda} G$$

such that the image of G consists of unitaries in $\widetilde{M}_G$, and the two images are related by the conjugation-equation $\pi(g.a) = \lambda(g)\pi(a)\lambda(g)^*$ (this can be easily written in diagrammatical terms). Define a Hilbert space

$$\widetilde{H}_G := \left\{\zeta \colon G \to H \mid \sum_{g\in G} \|\zeta(g)\|^2 < \infty\right\}$$

and embeddings into $B(\widetilde{H}_G)$ by $\pi(a)(\zeta) \colon g \mapsto (g^{-1}.a)\zeta(g)$ and $\lambda(g)(\zeta) \colon u \mapsto \zeta(g^{-1}.u)$; $M \curlyvee G$ is now defined to be the bicommutant $\big(\pi(M) \cup \lambda(G)\big)''$. Notice that $\widetilde{H}_G \cong H \otimes \ell^2(G)$.

categorification?

1. Considering G as a one-object premonoidal category, say $G[1]$, a G-action on M is a functor $G \to M$; $\Rightarrow$ consider the category $\mathrm{Funct}(G, \mathbf{C})$ for a Von Neumann category $\mathbf{C} \le \mathbf{Hilb}_{\otimes H}$.
2. $\widetilde{H}_G$ defined above is isomorphic to the tensor product $H \otimes \ell^2(G) = \{f \colon G \to \mathbb{C}$ such that $\sum_g |f(g)| < \infty\}$; then a basis of $\widetilde{H}_G$ is given by $\mathscr{B} = \big\{e_i \otimes \delta_g\big\}_{i \in I, g \in G}$, $\{e_i\}$ being a basis of H and $\{\delta_g\}$ is the basis of $\ell^2(G)$ made by Kronecker's deltas.
3. Define $\mathcal{L} \colon G[1] \to \mathbf{Hilb}_{\otimes \widetilde{H}_G} \leftarrow \mathbf{C} \colon \mathcal{P}$. The strong premonoidal functor $\mathcal{L}$ amounts to a map $G \to B(\widetilde{H}_G)$ which can be defined on the basis $\mathscr{B}$ as

$$\mathcal{L}(g) \colon e_i \otimes \delta_h \mapsto e_i \otimes \delta_{gh}.$$

4. $\mathcal{P} \colon \mathbf{C} \to \mathbf{Hilb}_{\otimes \widetilde{H}_G}$ acts as the identity on objects and sends $f \colon X \otimes \widetilde{H}_G \to Y \otimes \widetilde{H}_G$ in **Hilb** to

$$\mathcal{P}(f) \colon x \otimes e_i \otimes \delta_g \mapsto \Big((g^{-1}.f)(x \otimes e_i)\Big) \otimes \delta_g.$$

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Let $\mathbf{C}$ a Von Neumann category, and G a discrete group regarded as a one-object premonoidal category. Let $\mathbf{A}(\mathbf{C}, G)$ be the union of the essential images of the functors $\mathcal{P}, \mathcal{L}$ defined above; then the *premonoidal crossed product* of $\mathbf{C}$ and G is defined to be $\mathbf{C} \curlyvee G := \mathbf{A}(\mathbf{C}, G)''$.

Open Questions

- Can the category **Loc** be turned into a site with respect to a Grothendieck topology J, in such a way that one can consider the topos of sheaves $\mathrm{Sh}(\mathbf{Loc}, J)$? Is there a (phisically sensible) way to do this? Is this topos phisically meaningful?

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- A local net of C^*-premonoidal categories is a presheaf of categories $K \to \mathrm{pC}^*\text{-}\mathbf{Cat}$, sending $i \colon U \hookrightarrow V$ into $i_* \colon \mathcal{A}(U) \hookrightarrow \mathcal{A}(V)$, where all the restriction natural transformation $(ji)_* \cong j_* i_*$ are the identity:
  - Is this fibered category a prestack/stack?
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  - What if we consider the more general case where $(ji)_* \cong j_* i_*$ is *not* the identity?
- Is it possible to establish a completely parallel theory of Von Neumann categories (double commutant theorem, theory of factors, classification of all VN subcategories of $\mathbf{Hilb}_{\otimes H} \dots$)

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