Vertical categorification
of classical AQFT.

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Introduction. Classical AQFT can be defined as a *cosheaf* $\mathcal{A}$ of C*-algebras, defined on the manifold of space-time (or more generally on a suitable lorentzian manifold playing such rôle) M, satisfying two axioms:

- (locality condition) for any two open sets $U, V \subseteq M$ such that $U \subseteq V$, the algebras $A(U), A(V)$ are in the same inclusion relation. Physically speaking this means that observables in an open region are *a fortiori* observables *in any superset* of that region; from a sheaf-theoretic point of view this amounts to impose a flabbiness condition to the functor $\mathcal{A}$.

- (Einstein causality) If U, V are spacelike separated regions, then $\mathcal{A}(U)$ and $\mathcal{A}(V)$ pairwise commute in the *quasilocal algebra* $\mathcal{A}^\circ = \mathcal{A}(M) = \varinjlim_{U\subseteq M} \mathcal{A}(U)$.

Now what if we want to suitably *categorify* this notion, extending it to the realm of tensor categories (that is, categories equipped with a tensor $\otimes$ subject to suitable axioms)? Thanks to **[Coecke]**, the process of reformulation of classical Quantum Mechanics (which happens to be localized at the category **Hilb** of complex Hilbert spaces) in the classical language of *compact closed dagger categories* and C*-*categories* can be given a deep motivation.

A C*-*category* is, roughly speaking, a category $\mathbf{C}$ enriched over the symmetric monoidal category of (complex) Banach *-algebras, such that for any $X \in \mathbf{C}$ the set $\hom_{\mathbf{C}}(X, X)$ is a *unital* C*-algebra: see **[Warner]**, ch. **15** for the precise Definition. The class of all C*-categories becomes a (2-)category if we define 1-cells $\mathbf{C} \to \mathbf{D}$ to be the collection of all *-*functors* $\mathcal{F} : \mathbf{C} \to \mathbf{D}$, and 2-cells to be *bounded* natural transformations $\mathcal{F} \to \mathcal{G}$.

The problem of the categorification of AQFTs is strictly linked with the problem of categorifying Einstein causality. In our main reference **[Comeau]** proposes to model the theory in such a way that Einstein causality corresponds to an higher-categorical analogue of the notion of *Von Neumann algebra*, a subalgebra $A \leq B(H)$ which equals its *double commutant* A'' (see the first section of **[Halvorson]**, or the *verbatim* transcription in Section **3.1**).

This higher-categorical counterpart is a Von Neumann *category*, where a suitable notion of categorical *commutant* plays the rôle of the set-theoretic one. A tensor subcategory of a tensor (premonoidal) category $(\mathbf{C}, \otimes, I)$ is "Von Neumann" precisely if it equals its (suitably defined) double commutant $\mathbf{A}''$.

This simple idea is founded on two cornerstones:

- The notion of *binoidal category*, where instead of a functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ playing the role of a tensor product, we have two collections of functors, R_A, L_B, which respectively behave like "right tensor product with A" and "left tensor product with B", for any $A, B \in \mathrm{Ob}_{\mathbf{C}}$. In the opinion of **[Comeau]**, this notion captures the Einstein causality condition in the new setting[1].

- The notion of *vertical categorification*, which serves (in the words of John Baez) as a tool to "find category-theoretic analogs of set-theoretic concepts by replacing *sets* with *categories*, *functions* with *functors*, and *equations* between functions by *natural isomorphisms* between functors, which in turn should satisfy certain equations of their own, called *coherence laws*." We establish to use the term *categorification* without providing a formal definition of it: The interested reader may refer to **[Baez]**'s review for a huge amount of evocative examples and unexpected connections between areas of Mathematics.

1 Monoidal and Premonoidal Categories.

1.1 Tensors, braidings and dualities.

Tensors. Let $\mathbf{C}$ be a category. A *tensor* in $\mathbf{C}$ consists of a covariant bifunctor $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C} \colon (V, W) \mapsto V \otimes W$. Bifunctoriality of $- \otimes -$ can be easily translated into the two equalities

$$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g') \tag{1}$$

$$1_V \otimes 1_W = 1_{V \otimes W} \tag{2}$$

valid for any 4-uple of composable arrows f, f', g, g' and any two objects V, W.

A (strict) *monoidal category* consists of a category $\mathbf{C}$ with a tensor $\otimes$, in which we can find a distinguished object $\mathbb{I}$, to be called *unit object* such that

$$V \otimes \mathbb{I} = V = \mathbb{I} \otimes V \qquad (\forall V \in \mathrm{Ob}_{\mathbf{C}})$$

and such that for any three U, V, W one has $U \otimes (V \otimes W) = (U \otimes V) \otimes W$.

[1] The best way to be more precise is to present the words of the author: "In quantum teleportation, for example, the two participants must pass a classical message. So when this occurs, they cannot be spacelike separated. We believe that an appropriate modification of AQFT would allow for such modelling. More specifically, one should associate some sort of category of local protocols to each region in spacetime. But what structure should the category have? A reasonable first guess would be that of a compact closed dagger category. But this leaves open the question of how to express Einstein Causality. We propose here modifying the usual notion of compact closed dagger category by replacing the monoidal structure with premonoidal structure, as introduced by Power and Robinson. One of the fundamental aspects of monoidal structure in a category is the bifunctoriality of the tensor product."

Strict monoidal categories are rather rare structures: even the archetypal example of the category $\mathbf{Mod}(R)$ of modules over a commutative unital ring R is far from being strict, because the isomorphism $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$, albeit being canonical, is not the identity map; one gets hence used to manage a slightly weaker notion, where instead of equalities one has *canonical isomorphisms*

$$
\begin{aligned}
V \otimes \mathbb{I} \cong V \cong \mathbb{I} \otimes V \qquad & \forall V \in \mathrm{Ob}_{\mathbf{C}} \\
U \otimes (V \otimes W) \cong (U \otimes V) \otimes W, \qquad & \forall U, V, W \in \mathrm{Ob}_{\mathbf{C}}
\end{aligned}
$$

in such a way that the (tri)natural isomorphism $\boldsymbol{\alpha}$, called *associator*, satisfies the following *pentagon identity*: the diagram

$$
\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha"] \arrow[d, "\alpha \otimes 1"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha"'] & A \otimes ((B \otimes C) \otimes D) \arrow[ur, "1_A \otimes \alpha"'] &
\end{tikzcd}
\tag{3}
$$

commutes. Considering strict monoidal categories is not a "real" generalization, as long as you share the categorical viewpoint of identyfing equivalent categories, in view of *Mac Lane coherence Theorem* (see **[Leinster]** for an example on how *not* to prove it):

Theorem 1.1 : *Any monoidal category is equivalent to a strict one.*

Braidings. Tensor product of (bi)modules over a commutative unital ring R is itself "commutative" up to isomorphism, in the sense that one can find a canonical involution $\sigma_{VW} \colon V \otimes W \to W \otimes V$; this is obviously defined extending by linearity the function $v \otimes w \mapsto w \otimes v$ defined on the generators of $V \otimes W$. It is clear that for any three R-modules U, V, W one has

$$
\begin{aligned}
\sigma_{U, V \otimes W} &= (1_V \otimes \sigma_{UW}) \circ (\sigma_{UV} \otimes 1_W) \\
\sigma_{U \otimes V, W} &= (\sigma_{UW} \otimes 1_V) \circ (1_U \otimes \sigma_{VW})
\end{aligned}
\tag{4}
$$

Axiomatizing these properties naturally leads us to the definition of a *braiding* and a *twist* in a monoidal category.

Definition 1.1. A *braiding* in a monoidal category $\mathbf{V}$ consists of a natural isomorphism

$$\boldsymbol{\sigma} \colon \otimes \longrightarrow \otimes \circ T \tag{5}$$

where $T \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ is the unique functor such that $\pi_1 \circ T = \pi_2$, $\pi_2 \circ T = \pi_1$, for $\pi_i \colon \mathbf{C} \times \mathbf{C}$ the canonical projection. For any triple of objects in $\mathbf{V}$ hence one has the identities (4). Any pair $(\mathbf{V}, \boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ is a braiding in a monoidal category $\mathbf{V}$, is called *braided monoidal* category (BMC for short).

Definition 1.2. A *twist operator* (simply twist, for short) in a braided monoidal category consists of a family of isomorphisms

$$
\begin{tikzcd}
U \arrow[r, "\theta_U"] \arrow[d, "f"'] & U \arrow[d, "f"] \\
V \arrow[r, "\theta_V"'] & V
\end{tikzcd}
$$

$$\boldsymbol{\theta} = \{\theta_V \colon V \to V\}, \tag{6}$$

such that for any two objects V, W one has

$$\theta_{V\otimes W} = \sigma_{WV} \circ \sigma_{VW} \circ (\theta_V \otimes \theta_W). \tag{7}$$

Naturality of $\boldsymbol{\theta}$ amounts to the equality $\theta_V \circ f = f \circ \theta_U$ for any $f \colon U \to V$, as shown in the diagram aside.

Notice that $\theta_{\mathbb{I}} = 1_{\mathbb{I}}$, a relation implied by invertibility of $\theta_{\mathbb{I}}$ and by the equality

$$\theta_{\mathbb{I}} \circ \theta_{\mathbb{I}} = (\theta_{\mathbb{I}} \otimes 1_{\mathbb{I}}) \circ (1_{\mathbb{I}} \otimes \theta_{\mathbb{I}}) = \theta_{\mathbb{I}} \otimes \theta_{\mathbb{I}} = \theta_{\mathbb{I}}. \tag{8}$$

Dualities. *Duality* in a monoidal category is a generalization of the same notion in $\mathbf{C} = \mathbf{Mod}(K)$ (K any field); in such a case the definition is given in terms of a nondegenerate bilinear pairing between a vector space V and its *dual* $V^* = \hom(V, K)$.

Rather than looking for a generalization of this precise idea, one prefers to axiomatize the presence of a pair of arrows $V^* \otimes V \to \mathbb{I}$, $\mathbb{I} \to V \otimes V^*$, called *valuation* and *covaluation*.

Definition 1.3. Let $\mathbf{V}$ be a monoidal category. Suppose that to any object $V \in \mathbf{V}$ one can associate another object V^* in a functorial way, and two arrows

$$b_V \colon \mathbb{I} \to V \otimes V^* \qquad d_V \colon V^* \otimes V \to \mathbb{I}. \tag{9}$$

The triple (V^*, b_V, d_V) is called a *duality* in $\mathbf{V}$ if the following identities are true:

$$\begin{aligned} (1_V \otimes d_V) \circ (b_V \otimes 1_V) &= 1_V \\ (d_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes b_V) &= 1_{V^*} \end{aligned} \tag{10}$$

$$
\begin{tikzcd}
\mathbb{I} \arrow[r, "b_V"] \arrow[d, "b_V"'] & V \otimes V^* \arrow[d, "\theta_V \otimes 1_{V^*}"] \\
V \otimes V^* \arrow[r, "1_V \otimes \theta_{V^*}"'] & V \otimes V^*
\end{tikzcd}
$$

(if $\mathbf{C}$ is strict, one can identify $V \otimes \mathbb{I}$ and V; otherwise the composition with the unitor isomorphism is implied). Notice that $V \mapsto V^*$, albeit being functorial, is not supposed to be an involution; we call a duality $(-)^*$ *compatible* with the braiding $\boldsymbol{\sigma}$ and the twist $\boldsymbol{\theta}$ if and only if for any $V \in \mathrm{Ob}_{\mathbf{V}}$ one has

$$(\theta_V \otimes 1_{V^*}) \circ b_V = (1_V \otimes \theta_{V^*}) \circ b_V. \tag{11}$$

Remark 1 : *The valuation and covaluation maps can be obviously regarded as natural transformations between the constant functor on* $\mathbb{I}$ *and* $\otimes \circ (1_{\mathbf{V}} \times (-)^*)$, $\otimes \circ ((-)^* \times 1_{\mathbf{V}})$ *in such a way that a duality is uniquely determined by the triple* $((-)^*, \boldsymbol{b}, \boldsymbol{d})$.

Definition 1.4. A *ribbon category* is a twisted bmc with a *compatible* duality $((-)^*, \boldsymbol{b}, \boldsymbol{d})$.

Remark 2 : *To any ribbon category one can associate its* mirror, *defined as the same category where the braiding and the twist* $\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\theta}}$ *are defined by*

$$\overline{\sigma}_{VW} = \sigma^{-1}_{VW}, \qquad \overline{\theta}_V = \theta^{-1}_V. \tag{12}$$

Traces and Dimensions. The definition of ribbon category is given in order to categorify the two notions of *trace* of an endomorphism $f\colon V \to V$ and *dimension* of an object (defined to be the trace of 1_V).

Let $\mathbf{V}$ be a ribbon category, and denote $K = \mathrm{End}(\mathbb{I})$ the set of endomorphisms of the unit object; K is a commutative monoid in an obvious way, with respect to composition of arrows (commutativity follows from the *Eckmann-Hilton* argument:

$$h \circ k = (h \otimes 1_{\mathbb{I}}) \circ (1_{\mathbb{I}} \otimes k) = h \otimes k = (1_{\mathbb{I}} \otimes k) \circ (h \otimes 1_{\mathbb{I}}) = k \circ h). \tag{13}$$

The trace of $f\colon V \to V$ and the dimension fo V will turn out to be *K-valued invariants* associated to V.

Definition 1.5 (Trace)**.** Let $(V, \sigma, (-)^*, \boldsymbol{b}, \boldsymbol{d})$ be a ribbon category, and $f\colon V \to V$ an endomorphism of $V \in \mathrm{Ob}_{\mathbf{V}}$. The *trace* of f, denoted $\mathrm{tr}(f) \in K = \mathrm{End}(\mathbb{I})$ is defined to be

$$\mathrm{tr}(f) := d_V \circ \sigma_{V,V^*} \circ ((\theta_V \circ f) \otimes 1_{V^*}) \circ b_V. \tag{14}$$

Proposition 1.1 : $\mathrm{tr}(f)$ *enjoys the following properties:*

- *For any pair of morphisms* $f\colon V \to W, g\colon W \to V$, *one has* $\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f)$.
- *For any pair of endomorphisms* $f\colon V \to V, g\colon W \to W$ *one has* $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \circ \mathrm{tr}(g)$.
- *For any* $k\colon \mathbb{I} \to \mathbb{I}$ *one has* $\mathrm{tr}(k) = k$.

Definition 1.6 (Dimension)**.** Let $V \in \mathrm{Ob}_{\mathbf{V}}$ be an object in a ribbon category. Define its *dimension* as $\dim V = \mathrm{tr}(1_V) \in K$.

Notice that

- Isomorphic objects have the same dimension;
- For any two $V, W \in \mathrm{Ob}_{\mathbf{V}}$ one has $\dim(V \otimes W) = \dim(V) \circ \dim(W)$;
- $\dim(\mathbb{I}) = 1 (= 1_{\mathbb{I}})$.

2 Premonoidal Categories.

Definition 2.1. A *binoidal category* consists of a category $\mathbf{C}$ endowed with two families of $\mathrm{Ob}_{\mathbf{C}}$-indexed endofunctors $\{R_A, L_A\}_{A \in \mathrm{Ob}_{\mathbf{C}}}$, such that $R_B(A) = L_A(B)$ for any $A, B \in \mathrm{Ob}_{\mathbf{C}}$.

The object $R_B(A) = L_A(B)$ is often denoted $A \otimes B$ and called the *binoidal product* of A, B; the correspondence $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is called *pretensor*. From now on we also write $R_B = - \otimes B$ and $L_A = A \otimes -$. The pretensor $\otimes$ is said to be *associative* if there exists an isomorphim $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ for any three objects A, B, C.

For any arrow $f\colon X \to Y$ in $\mathbf{C}$, we denote $L_A(f) = 1_A \otimes f$ and $R_B(f) = f \otimes 1_B$; the intuition behind the definition of a binoidal category is a monoidal category where the pretensor $\otimes$ is not *bi*functorial, albeit being functorial when saturated in one of its two "arguments".

In a binoidal category there are in principle *two* different ways to compose a pair of arrows, and the tensor product of $f\colon A \to C$ and $g\colon B \to D$ is ambiguous as long as it is written "$f \otimes g$".

Definition 2.2 (Right and left product)**.** Suppose $(\mathbf{C}, \otimes)$ is binoidal, and define for any $f\colon A \to C, g\colon B \to D$ the *right* and *left product* of f and g, g and f, to be

$$g \rtimes f := (g \otimes 1_C) \circ (1_B \otimes f) \qquad\qquad g \ltimes f := (1_D \otimes f) \circ (g \otimes 1_A). \tag{15}$$

Definition 2.3 (Central Morphisms)**.** Suppose $(\mathbf{C}, \otimes)$ is binoidal, we say that $f\colon A \to C$ is *central* if for any $g\colon B \to D$ one has $g \rtimes f = g \ltimes f$ and $f \rtimes g = f \ltimes g$.

A natural transformation $\boldsymbol{\alpha}\colon \mathcal{G} \Rightarrow \mathcal{H}$ between functors $\mathcal{G}, \mathcal{H}\colon (\mathbf{B}, \otimes_{\mathbf{B}}) \to (\mathbf{C}, \otimes_{\mathbf{C}})$ is said to be central if every α_A is a central map.

One can easily notice that there is a link between centrality and bifunctoriality: more precisely the bifunctoriality of a pretensor $\otimes_{\mathbf{C}}$ can easily be translated into a diagrammatical form, and bifunctoriality as expressed in equation (1) precisely happens when *any* morphism f is central.

Centrality for $f\colon A \to C$ can be easily restated by asking that for any $g\colon B \to D$ the two squares

$$\begin{tikzcd} A \otimes B \arrow[r, "f\otimes B"] \arrow[d, "A\otimes g"'] \arrow[dr, shift left, "f\ltimes g"] \arrow[dr, shift right, "f\rtimes g"'] & C \otimes B \arrow[d, "C\otimes g"] \\ A \otimes D \arrow[r, "f\otimes D"'] & C \otimes D \end{tikzcd} \qquad \begin{tikzcd} B \otimes A \arrow[r, "B\otimes f"] \arrow[d, "g\otimes A"'] \arrow[dr, shift left, "g\ltimes f"] \arrow[dr, shift right, "g\rtimes f"'] & C \otimes B \arrow[d, "g\otimes C"] \\ D \otimes A \arrow[r, "D\otimes f"'] & C \otimes D \end{tikzcd} \tag{16}$$

commute. In case f is central we denote $f \otimes g = f \ltimes g = f \rtimes g$ and $g \otimes f = g \ltimes f = g \rtimes f$.

Definition 2.4. A *premonoidal category* (PMC for short) consists of a binoidal category $(\mathbf{C}, \otimes_{\mathbf{C}})$, where $\otimes = \otimes_{\mathbf{C}}$ is an associative pretensor, and an object $I \in \mathrm{Ob}_{\mathbf{C}}$ playing the rôle of a unit for the binoidal product, namely such that there are central natural equivalences

$$\begin{aligned} \alpha\colon (-\otimes-)\otimes- &\implies -\otimes(-\otimes-), \\ \lambda\colon -\otimes I &\implies \mathrm{id}_{\mathbf{C}}, \\ \varrho\colon I\otimes- &\implies \mathrm{id}_{\mathbf{C}} \end{aligned} \tag{17}$$

satisfying the exact formal analogue of the coherence conditions for a monoidal category.

A premonoidal category is said to be *symmetric* if there exists a central natural equivalence with components $\tau_{AB}\colon A \otimes B \to B \otimes A$, analogous to the symmetry of the monoidal case

REMARK 3 : *The following remark is taken almost* verbatim *from [Power].*

The aim of this paragraph is to better understand the notion of binoidal category, if possible expressing it in terms of more elementary notions.

First of all recall that the (2-)category **Cat** *is a* cosmos *in the sense of Bénabou, i.e. a complete and cocomplete closed symmetric monoidal category, with respect to the "cartesian product of categories" tensor and where the internal-hom is given exactly by the category of functors between two fixed categories.*

Conversely, it is not so well known that **Cat** *admits* exactly one *different cosmos structure, where the tensor is given by the category* $\mathbf{C}\#\mathbf{C}$ *having the same objects as* $\mathbf{C}\times\mathbf{C}$ *and where the set of morphisms between* (C,D) *and* (C',D') *is given by the set of "directed paths" with a suitable composition law*[2].

Now one can define a binoidal category as an internal magma *in* $\mathbf{Cat}_\#$ *(=the (2-)category* **Cat** *endowed with the* #*-symmetric monoidal structure), and a (strict) premonoidal category as a monoid in* $\mathbf{Cat}_\#$ *(in the same way a monoidal category was a monoid in* $\mathbf{Cat} = \mathbf{Cat}_\times$*).*

Example 2.1 : *Any monoidal category is obviously premonoidal, because* $\otimes\colon \mathbf{C}\times\mathbf{C}\to\mathbf{C}$ *is a bifunctor and* $f\ltimes g = g\otimes f = f\rtimes g$ *for any two arrows* f,g.

Example 2.2 : *Any monoid* $(M,\cdot)$ *can be regarded as a premonoidal category with a single object* $*$ *and such that* $\mathrm{End}(*) = M$, *i.e. where the tensor product amounts to multiplication of elements in* M*. This category is monoidal exactly when the operation is* commutative.

Example 2.3 : *Suppose* $\mathbf{C}$ *is a category, and denote* $[\mathbf{C},\mathbf{C}]$ *the category of endofunctors of* $\mathbf{C}$, *whose arrows* $\mathcal{F}\Rightarrow\mathcal{G}$ *are* transformations, *that is families of* $\mathrm{Ob}_{\mathbf{C}}$*-indexed arrows in* $\mathbf{C}$, $\{\eta_C\colon F(C)\to G(C)\}_{C\in\mathbf{C}}$, *without any further assumption. Tensor amounts to composition of functors; define* $(H\otimes\eta)_C = (H*\eta)_C = H(\eta_C)$, $(\eta\otimes H)_C = (\eta*H)_C = \eta_{H(C)}$*. Then* $[\mathbf{C},\mathbf{C}]$ *is a premonoidal category, monoidal if we restrict* $\hom_{[\mathbf{C},\mathbf{C}]}(F,G)$ *to the* natural *transformations between functors.*

We now define the notion of premonoidal functor: the central request that it preserves centrality of arrows, albeit natural, is not so *naïve*: see **[Power]** for more informations.

Definition 2.5 (Premonoidal Functor)**.** Let $(\mathbf{C},\otimes_{\mathbf{C}})$ and $(\mathbf{D},\otimes_{\mathbf{D}})$ be two binoidal categories. A *premonoidal functor* consists of a triple $(\mathcal{F},F_\otimes,F_1)$ where

- $\mathcal{F}\colon\mathbf{C}\to\mathbf{D}$ is a functor which sends central maps in $\mathbf{C}$ to central maps in $\mathbf{D}$;
- $F_\otimes$ consists of a family of natural arrows $F_{\otimes,CD}\mathcal{F}(C)\otimes_{\mathbf{D}}\mathcal{F}(D)\to\mathcal{F}(C\otimes_{\mathbf{C}} D)$

[2] The definition is of course possible in the case of two *different* categories $\mathbf{C},\mathbf{D}$; notice that $\mathbf{C}\#\mathbf{D}$ can be seen as a categorification of the free product of two monoids (=one-object categories). It is still open to me if $\mathbf{C}\#\mathbf{D}$ can be characterized as $\mathcal{F}(\mathcal{U}(\mathbf{C})\times\mathcal{U}(\mathbf{D}))$, where $\mathcal{F}\dashv\mathcal{U}$ is the adjoint pair "free category on a graph"$\dashv$ forgetful functor. If the explicit definition makes you feel uneasy, maybe you prefer to see $\mathbf{C}\#\mathbf{D}$ defined via a universal property, which is exactly the definition we gave for a binoidal category. $\mathbf{C}\#\mathbf{D}$ is the unique category $\mathbf{X}$ equipped with two families of functors $\{\mathcal{F}_C\colon\mathbf{D}\to\mathbf{X}\}_{C\in\mathrm{Ob}_{\mathbf{C}}},\{\mathcal{G}_D\colon\mathbf{C}\to\mathbf{X}\}_{D\in\mathrm{Ob}_{\mathbf{D}}}$ such that $\mathcal{F}_C(D) = \mathcal{G}_D(C)$ for any $(C,D)\in\mathrm{Ob}_{\mathbf{C}\times\mathbf{D}}$.

such that the following diagram commute:

$$\begin{tikzcd}
(\mathcal{F}(A)\otimes\mathcal{F}(B))\otimes\mathcal{F}(C) \arrow[r] \arrow[d] & \mathcal{F}(A)\otimes(\mathcal{F}(B)\otimes\mathcal{F}(C)) \arrow[d] \\
\mathcal{F}(A\otimes B)\otimes\mathcal{F}(C) \arrow[d] & \mathcal{F}(A)\otimes\mathcal{F}(B\otimes C) \arrow[d] \\
\mathcal{F}((A\otimes B)\otimes C) \arrow[r] & \mathcal{F}(A\otimes(B\otimes C))
\end{tikzcd} \tag{18}$$

- $F_1\colon J \to F(I)$ is an arrow such that

$$\begin{tikzcd}
\mathcal{F}(A)\otimes J \arrow[r] \arrow[d] & \mathcal{F}(A)\otimes\mathcal{F}(I) \arrow[d] \\
\mathcal{F}(A) & \mathcal{F}(A\otimes I) \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
J\otimes\mathcal{F}(A) \arrow[r] \arrow[d] & \mathcal{F}(I)\otimes\mathcal{F}(A) \arrow[d] \\
\mathcal{F}(A) & \mathcal{F}(I\otimes A) \arrow[l]
\end{tikzcd} \tag{19}$$

  commute. A premonoidal functor is said to be *strong*, resp. *strict* if $F_\otimes, F_1$ are isomorphisms, resp. identity maps. The dual notion of an *op-premonoidal functor* $(\mathcal{F}, {}_\otimes F, {}_1F)$ is the same, but with ${}_\otimes F_{CD}\colon \mathcal{F}(C\otimes_{\mathbf{C}} D) \to \mathcal{F}(C)\otimes_{\mathbf{D}} \mathcal{F}(D)$, ${}_1F\colon F(I) \to J$; op-strictness and op-strength are defined in the same way.

It is now a matter of analogy to define the notion of *premonoidal natural transformation*:

Definition 2.6 (Premonoidal natural transformation)**.** A *premonoidal natural transformation* between two premonoidal functors consists of a *central* transformation $\mathcal{F} \to \mathcal{G}$ which is natural as well as *compatible* with $F_\otimes, G_\otimes, F_1, G_1, {}_\otimes F, {}_\otimes G, {}_1F, {}_1G$: we can compactify the compatibility and op-compatibility condition expressing it via the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{F}(C)\otimes\mathcal{F}(D) \arrow[r, shift left, "F_{\otimes,CD}"] \arrow[d, "\eta_C\otimes\eta_D"'] & \mathcal{F}(C\otimes D) \arrow[l, shift left, "{}_\otimes F_{CD}"] \arrow[d, "\eta_{C\otimes D}"] \\
\mathcal{G}(C)\otimes\mathcal{G}(D) \arrow[r, shift left, "G_{\otimes,CD}"] & \mathcal{G}(C\otimes D) \arrow[l, shift left, "{}_\otimes G_{CD}"]
\end{tikzcd} \tag{20}$$

and a totally analogous one for the compatibility with $F_1, {}_1F$. Centrality is required as long as we want a unique way to get an arrow $\eta_C\otimes\eta_D\colon \mathcal{F}(C)\otimes\mathcal{F}(D) \to \mathcal{G}(C)\otimes\mathcal{G}(D)$.

Definition 2.7. Let $(\mathbf{C},\otimes)$ be a PMC. Its *center* $Z(\mathbf{C})$ consists of the subcategory with the same objects, where $\hom(A,B)$ is made by all central arrows between A and B.

Like the notion of monoidal category can be seen to (vertically) categorify the notion of monoid, the center of a premonoidal category categorifies the center $Z(M)$ of a monoid M. It's easy to notice that the center of a premonoidal category is a monoidal category (almost by definition).

Example 2.4 : *Let* $(\mathbf{C},\otimes,I)$ *a strict* monoidal *category with a simmetry* $\tau_{XY}\colon X\otimes Y\to Y\otimes X$*, and let* $S\in \mathrm{Ob}_{\mathbf{C}}$*. Define a category* $\mathbf{C}_{\otimes S}$ *with the same objects of* $\mathbf{C}$*, and where* $\hom_{\mathbf{C}_{\otimes S}}(X,Y)=\hom_{\mathbf{C}}(X\otimes S,Y\otimes S)$*. Define* $Z\otimes f=1_Z\otimes f$ *and* $f\otimes Z$ *to be*

$$X\otimes Z\otimes S\xrightarrow{\tau_{XZ}\otimes 1_S} Z\otimes X\otimes S\xrightarrow{1_Z\otimes f} Z\otimes Y\otimes S\xrightarrow{\tau_{ZY}\otimes 1_S} Y\otimes Z\otimes S. \tag{21}$$

With these definitions, $\mathbf{C}_{\otimes S}$ *is a premonoidal category: associativity and unit follow from the structure on* $\mathbf{C}$*.*

We now apply this definition to a particular useful case.

Proposition 2.1 : *Every central morphism* $f\in\hom_{\mathbf{Hilb}_{\otimes H}}(X,Y)$ *comes from* $\widehat{f}\in\hom_{\mathbf{Hilb}}(X,Y)$*, via* $f=\hat f\otimes 1_H$*.*

Proof. Choose an orthonormal basis $\mathscr{B}_H=\{h_j\mid j\in\mathcal{J}\}$. If $a\neq b\in\mathcal{J}$ define $T_{ab}\colon H\otimes_{\mathbb{C}} H\to H\otimes_{\mathbb{C}} H$ by

$$T_{ab}\Big(h_i\otimes h_j\Big)=\Big(\delta_{ia}\delta_{jb}+\delta_{ib}\delta_{ja}\Big)h_j\otimes h_i \tag{22}$$

that is $T_{ab}\colon h_i\otimes h_j\mapsto h_b\otimes h_a+h_a\otimes h_b$ is defined to be the projection on the subspace $(H\otimes H)_{ab}=\langle h_b\otimes h_a, h_a\otimes h_b\rangle$. Choose other bases

$$\mathscr{B}_X=\{e_i\mid i\in\mathcal{I}\}$$
$$\mathscr{B}_Y=\{g_k\mid k\in\mathcal{K}\}$$

and compute $f\rtimes T_{ab}$ on the basis $e_i\otimes h_j\otimes h_k$:

$$\begin{aligned}(f\rtimes T_{ab})(e_i\otimes h_j\otimes h_k)&=(f\otimes 1_H)(1_X\otimes T_{ab})(e_i\otimes h_j\otimes h_k)\\&=(f\otimes 1_H)\Big(\delta_{ia}\delta_{jb}+\delta_{ib}\delta_{ja}\Big)(e_i\otimes h_j\otimes h_k),\end{aligned}$$

if $j=k=a$ and $a\neq b$ one has $(f\rtimes T_{ab})(e_i\otimes h_j\otimes h_k)=0$.

Now let's compute $f\ltimes T_{ab}$ on the basis $e_i\otimes h_j\otimes h_k$, knowing that

$$\begin{aligned}(f\otimes 1_H)(e_i\otimes h_j\otimes h_k)&=(\tau\otimes 1_H)(h_j\otimes f(e_i\otimes h_k))\\&=(\tau\otimes 1_H)\Big(h_j\otimes \textstyle\sum_{r,p}c_{ik}^{rp}g_r\otimes h_p\Big)\\&=\textstyle\sum_{r,p}c_{ik}^{rp}g_r\otimes h_j\otimes h_p\end{aligned}$$

this entails $f\ltimes T_{ab}(e_i\otimes h_j\otimes h_k)=\sum_{r,p}c_{ik}^{rp}(\delta_{ja}\delta_{pb}+\delta_{pa}\delta_{jb})g_r\otimes h_p\otimes h_j$. If $j=k=a$, $a\neq b$, then

$$\sum_{r,p}c_{ik}^{rp}\delta_{pb}g_r\otimes h_p\otimes h_j=\sum_r c_{ia}^{rb}g_r\otimes h_b\otimes h_a; \tag{23}$$

now, if we use the fact that $f\ltimes T_{ab}=f\rtimes T_{ab}$ we notice that $\sum_r c_{ia}^{rb}g_r\otimes h_b\otimes h_a=0$, and hence that $c_{ia}^{rb}=0$ whenever $a\neq b$; from this, nad noticed also that

$$(f\rtimes T_{ab})(e_i\otimes h_j\otimes h_k)=\sum_{r\in\mathcal{K}}c_{ij}^{rj}(\delta_{ja}\delta_{kb}+\delta_{jb}\delta_{ka})g_r\otimes h_k\otimes h_j$$
$$(f\ltimes T_{ab})(e_i\otimes h_j\otimes h_k)=\sum_{r\in\mathcal{K}}c_{ik}^{rk}(\delta_{ja}\delta_{kb}+\delta_{jb}\delta_{ka})g_r\otimes h_k\otimes h_j$$

we can deduce that $c_{ij}^{rj}(\delta_{ja}\delta_{kb}+\delta_{jb}\delta_{ka}) = c_{ik}^{rk}(\delta_{ja}\delta_{kb}+\delta_{jb}\delta_{ka})$, which for $a = j, b = k$ entails

$$c_{ia}^{ra}(1+\delta_{ab}) = c_{ib}^{rb}(1+\delta_{ab}), \tag{24}$$

whence we deduce that $c_{ia}^{ra} = c_{ib}^{rb}$ for any $a, b \in \mathcal{J}$, and we can define $d_i^r := c_{ia}^{ra}$. We are almost done, because

$$f(e_i \otimes h_a) = \sum_{r\in\mathcal{K}} d_i^r g_r \otimes h_a = \Big(\sum_r d_i^r g_r\Big) \otimes h_a =: \hat{f}(e_i) \otimes h_a. \tag{25}$$

It remains only to notice that $\hat{f}$ is bounded, because $\|f\| = \|\hat{f} \otimes 1_H\| = \|\hat{f}\|$. □

3 Von Neumann Categories.

3.1 Von Neumann Algebras.

This brief introduction to Von Neumann algebras comes almost *verbatim* from the first pages of **[Halvorson]**'s review on AQFTs. Recall that if H is a complex Hilbert space, then the algebra $B(H)$ of continuous linear operators $H \to H$ can be endowed with different topologies: we consider

- The *uniform* topology, turning $B(H)$ into a Banach space, induced by the operator norm
$$\|T\|_{\mathrm{u}} := \sup_{|v|\le 1} |T(v)|. \tag{26}$$
An element $T \in B(H)$ is the limit of a sequence $\{T_i\} \subset B(H)$ if and only if the sequence $\{\|T - T_i\|_{\mathrm{u}}\} \subset \mathbb{R}$ converges to zero.

- The *weak* topology, defined via the family of seminorms $\{\mathrm{p}_{u,v} \mid u, v \in H\}$, $\mathrm{p}_{u,v}(T) := |\langle u, Tv\rangle|$; this last topology is in general not first countable, hence the closure of $S \subseteq B(H)$ can be obtained as the set of all limit points of S-valued *nets*. A net $\{T_\lambda\}$ converges strongly to $T \in B(H)$ if and only if $\{\mathrm{p}_{u,v}(T_\lambda)\}$ converges to $\mathrm{p}_{u,v}(T)$ in $\mathbb{R}$, for any $u, v \in H$.

- The *strong* topology, defined via the family of seminorms $\{\mathrm{p}_v \mid v \in H\}$, $\mathrm{p}_v(T) := |Tv|$; a net $\{T_\lambda\}$ converges strongly to $T \in B(H)$ if and only if $\{\mathrm{p}_v(T_\lambda)\}$ converges to $\mathrm{p}_v(T)$ in $\mathbb{R}$, for any $v \in H$.

These three topologies are ordered by the chain

$$\text{weak} \preceq \text{strong} \preceq \text{uniform}. \tag{27}$$

It's useful to notice that all three topologies coincide if H is finite dimensional, and that the closures of any bounded, convex subset $\mathcal{S} \subseteq B(H)$ coincide in the weak, strong and uniform topology.

Definition 3.1 (Commutant)**.** Let $\mathcal{A} \subset B(H)$ a subalgebra closed under conjugation; define the *commutant* of A as

$$\{T \in B(H) \mid TX = XT \ \forall X \in \mathcal{A}\} \tag{28}$$

and denote it as $\mathcal{A}'$.

Example 3.1 : Schur's lemma *stated in the language of commutants says that if* $\varrho\colon G \curvearrowright H$ *is an irreducible unitary representation of a topological group, then* $(\mathrm{im}\varrho)' \cong \mathbb{C}$.

Remark 4 : *If we consider the set* $\mathfrak{A}(H)$ *of all conjugate-closed subalgebras of* $B(H)$ *then* $\mathcal{A} \mapsto \mathcal{A}'$ *defines a correspondence* $\mathfrak{A}(H) \to \mathfrak{A}(H)$*, which composed twice turns out to act as a closure operator on* $\mathfrak{A}(H)$ *if we partial-order it with respect to inclusion: it means that* $\mathcal{A} \subseteq \mathcal{A}'' =: \mathrm{cl}(\mathcal{A}) = \mathrm{cl}(\mathrm{cl}(\mathcal{A}))$ *and* $\mathrm{cl}(\mathcal{A} \cap \mathcal{B}) = \mathrm{cl}(\mathcal{A}) \cap \mathrm{cl}(\mathcal{B})$.

This is not so surprising, because of the intimate link between double commutants (i.e conugate-closed algebras $\mathcal{A} \subseteq B(H)$ such that $\mathrm{cl}(\mathcal{A}) = \mathcal{A}$) and weakly closed unital subalgebras of $B(H)$, summarized by the following theorem:

Theorem 3.1 [Von Neumann's double commutant]: *Let* $\mathcal{A} \subseteq B(H)$ *a conjugate-closed unital subalgebra; then it is Von Neumann iff it is weakly closed.*

3.2 Premonoidal *-structures and Von Neumann categories.

Let's collect in a single place various definitions about *-premonoidal categories: it shouldn't be surprising that they are the exact formal analogues of those in the monoidal case (see **[Comeau2]**, §**5.5**):

Definition 3.2. A premonoidal category is said to be $\mathbb{C}$-*linear* if it is enriched over $\mathbf{Vect}_{\mathbb{C}}$; a *positive* **-operation* on a $\mathbb{C}$-linear premonoidal category $\mathbf{C}$ consists of an antiequivalence $(-)^*\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ such that

- It is the identity on objects and an antilinear map on the level of morphisms;
- $(-)^{**} = \mathrm{id}_{\mathbf{C}}$ and $1_X = 1_X^*$ for any $X \in \mathrm{Ob}_{\mathbf{C}}$;
- for any $f\colon X \to Y$, the two arrows $f^* \circ f\colon X \to X$ and $f \circ f^*\colon Y \to Y$ are the zero map iff $f = 0_{XY}$.

A *premonoidal* **-category* consist of a $\mathbb{C}$-linear premonoidal category endowed with a positive *-operation: . Finally a *premonoidal* C**-category* $(\mathbf{C}, \otimes, I, \|-\|_\bullet)$ consists of a premonoidal Banach-*-algebra-enriched $\mathbb{C}$-linear category, such that

$$\|g \circ f\|_{XZ} \le \|g\|_{YZ} \cdot \|f\|_{XY}, \quad \|f^* \circ f\|_X = \|f \circ f^*\|_Y = \|f\|^2_{XY} \tag{29}$$

for any two morphisms of $\mathbf{C}$ $f\colon X \to Y$, $g\colon Y \to Z$.

It is straightforward to notice that the center of a premonoidal C*-category is a C*-category in the sense of **[Warner]**, §**15**.

The aim of the rest of this subsection is to unravel the following crude definition:

> The notion of *commutant* $\mathcal{A}'$ with which we defined Von Neumann algebras can be easily categorified into the notion of commutant in a *Von Neumann category*, which is a subcategory of a C*-category $\mathbf{C}$ such that $\mathbf{A}''(X, Y) = \mathbf{A}(X, Y)$.

Definition 3.3. Let $\mathbf{A} \subseteq \mathbf{C}$ be a subcategory of a premonoidal *-category $\mathbf{C}$; the *commutant* of $\mathbf{A}$, denoted $\mathbf{A}'$, is the subcategory of $\mathbf{C}$ with the same class og objects and having as $\mathbf{A}'(X,Y)$ the set of all $f\colon X \to Y$ such that $f \ltimes g = f \rtimes g$, $g \ltimes f = g \rtimes f$ for all $g \in \mathrm{Mor}(\mathbf{A})$.

Theorem 3.2 : *The commutant $\mathbf{A}'$ of $\mathbf{A} \subseteq \mathbf{C}$ is a *-premonoidal category provided $\mathbf{A}$ is such a category.*

Proof. The most important thing to check is that composition of arrows in $\mathbf{A}'$ remains in $\mathbf{A}'$: suppose $f\colon A \to B$, $h\colon B \to C$ commute with every $g\colon X \to Y$ in $\mathbf{A}$. Then for such a g one has (denote by mere juxtaposition the composition between two arrows)

$$\begin{aligned}(hf) \ltimes g &= (1_C \otimes g)(hf \otimes 1_X)\\ &= (1_C \otimes g)(h \otimes 1_X)(f \otimes 1_X)\\ &= (h \otimes 1_Y)(1_B \otimes g)(f \otimes 1_X)\\ &= (hf \otimes 1_Y)(1_A \otimes g) = (hf) \rtimes g\end{aligned}$$

Similarly one checks that $g \ltimes hf = g \rtimes hf$, hence $h \circ f \in \mathbf{A}'(A,C)$. Clearly this composition law fits all the axioms turning $\mathbf{A}'$ into a category.

The premonoidal structure on $\mathbf{A}$ induces by restriction a premonoidal structure, because given $f \in \mathbf{A}'(X,Y)$, $g \in \mathbf{A}(A,B)$, then for any $Z \in \mathbf{A}$

$$\begin{array}{rcl}(1_Z \otimes f) \ltimes g &=& (1_Z \otimes f) \rtimes g\\ g \ltimes (1_Z \otimes f) &=& g \rtimes (1_Z \otimes f)\end{array} \tag{30}$$

Let's show for example that $(1_A \otimes f) \ltimes g = (1_A \otimes f) \rtimes g$, any other check being analogous (but pretty boring):

$$\begin{aligned}(1_Z \otimes f) \ltimes g &= (1_Z \otimes 1_Y \otimes g) \circ (1_Z \otimes f \otimes 1_A)\\ &= 1_Z \otimes \big[(1_Y \otimes g) \circ (f \otimes 1_A)\big]\\ &= 1_Z \otimes \big[(f \otimes 1_B) \circ (1_X \otimes g)\big]\\ &= [(1_Z \otimes f) \otimes 1_B] \circ [1_{Z \otimes X} \otimes g] = (1_Z \otimes f) \rtimes g.\end{aligned}$$

Since every coherence condition (associativity and unit diagrams) involves arrows living in the center $Z(\mathbf{C}) \subseteq \mathbf{A}'$. We conclude noticing that $(-)^*$ is a tensor functor, namely it commutes with any functor $A \otimes -$, $- \otimes B$. □

We are now ready to give the very definition of Von Neumann category:

Definition 3.4. Let $\mathbf{A} \subseteq \mathbf{C}$ be a premonoidal C*-subcategory of a premonoidal C*-category $\mathbf{C}$; then $\mathbf{A}$ is called a $\mathbf{C}$-*Von Neumann category* if $\mathbf{A}''(X,Y) = \mathbf{A}(X,Y)$ for any $X, Y \in \mathrm{Ob}_{\mathbf{A}}$. When the context is clear, or when $\mathbf{C} = \mathbf{Hilb}_{\otimes H}$, a $\mathbf{C}$-Von Neumann category is simply said *Von Neumann*.

As an example of this step-by-step categorification, let the following example show that the new theory really embodies the old one:

Proposition 3.1 : *Suppose $\mathbf{A}$ is a Von Neumann category; then $\mathbf{A}(\mathbb{C},\mathbb{C})$ is a Von Neumann algebra in the sense of Definition 3.1.*

Proof. By definition, $\mathbf{A}(\mathbb{C},\mathbb{C}) = \mathcal{M}$ is a *-subalgebra of $B(\mathbb{C}\otimes H) \cong B(H)$, hence its elements are bounded operators on H. So it's easy to link $S \in \mathcal{M}$ to the classical commutant of operators: suppose $S \in \mathbf{A}'(\mathbb{C},\mathbb{C})$ and $T \in \mathcal{M}$; $S \ltimes T = S \rtimes T$ means that the diagram

$$\begin{tikzcd}
\mathbb{C}\otimes\mathbb{C}\otimes H \arrow[rr, "1_{\mathbb{C}}\otimes S"] \arrow[d, "\tau\otimes 1_H"'] & & \mathbb{C}\otimes\mathbb{C}\otimes H \arrow[r, "\tau\otimes 1_H"] & \mathbb{C}\otimes\mathbb{C}\otimes H \arrow[d, "1_{\mathbb{C}}\otimes T"] \\
\mathbb{C}\otimes\mathbb{C}\otimes H \arrow[r, "1_{\mathbb{C}}\otimes T"'] & \mathbb{C}\otimes\mathbb{C}\otimes H \arrow[r, "\tau\otimes 1_H"'] & \mathbb{C}\otimes\mathbb{C}\otimes H \arrow[r, "1_{\mathbb{C}}\otimes S"'] & \mathbb{C}\otimes\mathbb{C}\otimes H
\end{tikzcd} \tag{31}$$

commutes. Once noticed that $\tau = \tau_{\mathbb{C}\mathbb{C}} = 1_{\mathbb{C}\otimes\mathbb{C}}$, this boils down to

$$(1_{\mathbb{C}} \otimes T) \circ (1_{\mathbb{C}} \otimes S) = (1_{\mathbb{C}} \otimes S) \circ (1_{\mathbb{C}} \otimes T) \tag{32}$$

hence to $T \circ S = S \circ T$. □

Corollary 1 : *Every one-object Von Neumann category is a Von Neumann algebra.*

One of the first properties of the commutant of a *-subalgebra $A \subseteq B(H)$ is that $A''' = A'$, so that the commutant of a *-closed subalgebra of $B(H)$ is a Von Neumann algebra, or in other words that the commutant (resp., double commutant) acts as a preclosure (resp., closure) operator in the sense of Kuratowski on the collection of *-subalgebras of C.

This very result has a formally identical analogue in the language of Von Neusmann categories:

Proposition 3.2 : *If* $\mathbf{A}$ *is a subcategory of a premonoidal* C**-category such that the functor* $(-)^*$ *restricts to a functor* $(-)^* \colon \mathbf{A}\times\mathbf{A} \to \mathbf{A}$, *then* $\mathbf{A}'$ *is a premonoidal* C**-category, and in particulat a* $\mathbf{C}$*-Von Neumann category.*

Proof. We already know that $\mathbf{A}'$ is a premonoidal C*-category. In fact, we only have to show that each $\mathbf{A}'(X,Y)$ is complete with respect to the norm it inherits from $\mathbf{C}(X,Y)$ (recall that there's only one admissible norm, once you're given the C*-structure to a Banach algebra).

For any $f\colon A \to B$ and $C,D \in \mathrm{Ob}_{\mathbf{C}}$ the linear map $\zeta(g,f)\colon \mathbf{C}(C,D) \to \mathbf{C}(A\otimes C, B\otimes D)$, defined by $(g,f) \mapsto f \ltimes g - f \rtimes g$ is (jointly in both arguments) bounded. Hence given a Cauchy sequence in $\mathbf{A}'(B,D)$, say $\{g_j\}$, by completeness it must converge to a suitable $g \in \mathbf{C}(B,D)$; now for any $f\colon A \to C$ in $\mathbf{A}$ one has

$$\zeta(g,f) = \zeta(\lim_{j\to\infty} g_j, f) = \lim_{j\to\infty} \zeta(g_j, f) = 0 \tag{33}$$

and in the same way $\zeta(f,g_j) = 0$, hence $g = \lim_{j\to\infty} g_j \in \mathbf{A}'(B,D)$.

To see that $\mathbf{A}'$ is a $\mathbf{C}$-Von Neumann category, we observe that $\mathbf{A} \subseteq \mathbf{A}''$, and taking commutants (which by definitions reverse inclusions) we get $\mathbf{A}''' \subseteq \mathbf{A}'$ for free. On the other hand, $\mathbf{A}' \subseteq \mathbf{A}'''$, and thus we conclude. □

Remark 5 : *Every premonoidal* C^**-category* $\mathbf{C}$ *admits two trivial Von Neumann subcategories, namely* $\mathbf{C}$ *itself and its center* $Z(\mathbf{C})$*; this is straightforward once we noticed that* $Z(\mathbf{C})' = \mathbf{C}$*, and then applying Proposition* ***3.2****.*

[Comeau] uses the above remark interpreting $\mathbf{Hilb}_{\otimes H}$ as a multi-object analogue of $B(H)$, and its center $Z(\mathbf{Hilb}_{\otimes H}) = \mathbf{Hilb}$ as a multi-object analogue of the ground field $\mathbb{C}$: this analogy can be illustrated via a diagram like this (see **[Comeau2]**, ch. **9.3**):

$$\begin{tikzcd}
B(H) \arrow[r, rightsquigarrow, "\text{categorification}"'] & \mathbf{Hilb}_{\otimes H} \\
M \arrow[u, hook] \arrow[r, rightsquigarrow, "\text{categorification}"'] & \mathbf{M} = \mathbf{M}'' \arrow[u, hook] \\
Z(M) \arrow[u, hook] \arrow[r, rightsquigarrow, "\text{categorification}"'] & Z(\mathbf{M}) \arrow[u, hook] \\
Z(B(H)) \cong \mathbb{C} \arrow[u, hook] \arrow[r, rightsquigarrow, "\text{categorification}"'] & \mathbf{Hilb} \cong Z(\mathbf{Hilb}_{\otimes H}) \arrow[u, hook]
\end{tikzcd} \tag{34}$$

where $M \subseteq B(H)$ on the left is a Von Neumann algebra and $\mathbf{M}$ on the right is a Von Neumann subcategory of $\mathbf{Hilb}_{\otimes H}$.

Example 3.2 : *Consider the category* $[\mathbf{C}, \mathbf{C}]_\star$ *of* **-endofunctors in a* C^**-category* $\mathbf{C}$*, having as arrows* bounded *transformations, namely collections of arrows* $\boldsymbol{t} = \{\mathcal{F}(A) \to \mathcal{G}(A)\}_{A \in \mathrm{Ob}_\mathbf{C}}$ *such that* $\|\boldsymbol{t}\| = \sup_{A \in \mathrm{Ob}_\mathbf{C}} \|t_A\| < \infty$*. Then* $\boldsymbol{t} \mapsto \|\boldsymbol{t}\|$ *defines a norm on* $[\mathbf{C}, \mathbf{C}]_\star$*, and with the tensor defined by*

$$\mathcal{F} \otimes \mathcal{G} := \mathcal{F} \circ \mathcal{G}$$
$$\mathcal{F} \otimes \boldsymbol{t} := \mathcal{F} * \boldsymbol{t} \qquad \boldsymbol{t} \otimes \mathcal{G} = \boldsymbol{t} * \mathcal{G}$$

the category $[\mathbf{C}, \mathbf{C}]_\star$ *becomes a premonoidal one, as described in Example* ***2.3****.*

Now, if we consider the subcategory $\mathbf{C}^\mathbf{C} \leq [\mathbf{C}, \mathbf{C}]_\star$ *made by endofunctors and* bounded natural *transformations, we have* $Z([\mathbf{C}, \mathbf{C}]_\star) \leq \mathbf{C}^\mathbf{C}$*, with proper inclusion.*

Indeed, consider a central map $\boldsymbol{t} \colon \mathcal{F} \to \mathcal{G}$*; then any* $f \colon A \to B$ *can be regarded as a transformation between constant functors on* A *and* B *respectively, let's denote it as* $\boldsymbol{s} \colon \Delta_A \to \Delta_B$*, where* $\Delta_A(X) = A, \Delta_B(Y) = B, s_X = f$ *for any* $X, Y \in \mathbf{C}$*; hence by centrality the diagram*

$$\begin{tikzcd}
\mathcal{F} \otimes \Delta_A(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "t_A = (\boldsymbol{t} * \Delta_A)_X"'] & \mathcal{F} \otimes \Delta_B(X) \arrow[d, "t_B = (\boldsymbol{t} * \Delta_B)_X"] \\
\mathcal{G} \otimes \Delta_A(X) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G} \otimes \Delta_B(X)
\end{tikzcd} \tag{35}$$

commutes for any $f \colon A \to B$*, which is exactly the naturality condition for* $\boldsymbol{t}$*. The converse inclusion is in general false: see Example* ***6.2.4*** *in* ***[Comeau2]****.*

4 Algebraic Quantum Field Theory.

The beginning of this section closely follows **[Brunetti]** and **[Comeau2]** in order to introduce in a self-contained way the basic definitions regarding Algebraic Quantum Field Theory; the rest of it is devoted to present one of the most elementary application of the theory built so far, that is the construction of *crossed products* of premonoidal C*-categories, as presented in **[Comeau]**, and to introduce the (completely new) material about categorification of AQFTs presented in **[Comeau2]**.

Roughly speaking an AQFT consists of a suitably well-behaved functor; more precisely, one can describe an AQFT as a functor between a suitable "category of regions" **Loc** whose objects are suitable subspaces of a given Lorentzian manifold modeling space-time, and whose morphisms are *(isometric) embeddings*, and a "category of observables" **Obs** whose objects are C*-algebras, modeling algebras of operators/observables on the spacetime, and whose morphisms are *(unital) embeddings*.

Algebraic QFTs live in the (2-)category Funct(**Loc**, **Obs**).

Definition 4.1 (Category of regions)**.** Let $d \geq 2$ be a fixed integer, and consider the category **Loc** whose objects are all smooth d-dimensional, globally hyperbolic, lorentzian, oriented and time-oriented manifolds (M,g) (more often denoted simply by M); for any two such M,N, the set $\mathbf{Loc}(M,N)$ consists of all the isometric embeddings $\iota\colon M \to N$ subject to the following constraints:

- Defining a *causal curve* as a curve $\gamma\colon [0,1] \to N$ such that $g(\dot\gamma(t),\dot\gamma(t)) < 0$, the subspace $M \cong \iota M \subseteq N$ is *causally convex*, meaning that it contains every causal curve whose endpoints are in ιM. These are called *causally convex* embeddings.

- Any morphism $\iota\colon M \to N$ preserves orientation and time-orientation of the embedded spacetime (refer to **[Penrose]** for a definition of time-orientation on a lorentzian manifold).

The composition of arrows in **Loc** amounts to composition of embeddings; the identity map $M \to M$ is the identity in $\mathbf{Loc}(M,M)$.

Definition 4.2 (Category of observables)**.** The category **Obs** of observables is formed by all unital complex C*-algebras, and $\mathbf{Obs}(C,D)$ are the injective unital *-morphisms $C \to D$; the composition amounts to composition of maps.

Definition 4.3 ((Locally covariant) QFT)**.** A *locally covariant Quantum Field Theory* is an object of the (2-)category $\mathcal{A} \in \mathrm{Funct}(\mathbf{Loc},\mathbf{Obs})$, often denoted $[\mathbf{Loc},\mathbf{Obs}]$ for short.

A locally covariant QFT is called *causal* if whenever $\iota_1(M_1) \subset N, \iota_2(M_2) \subset N$ are *causally separated*, i.e. there exists no causal curve between them, then $[A_1,A_2] = 0$ for any $A_1 \in \mathcal{A}(M_1), A_2 \in \mathcal{A}(M_2)$ and both are considered as subalgebras of $\mathcal{A}(N)$.

The *generally covariant locality* principle amounts to say that to any globally hyperbolic lorentzian spacetime M can be assigned a C*-algebra $\mathcal{A}(M) = C$ such that the corresponding algebras can be identified when two given spacetimes are isometrically isomorphic.

Notice that according to Definition **4.3** a locally covariant QFT is in principle *not* a C^*-**Alg**-valued presheaf on some "total" lorentzian manifold M, unless one can impose rather strong size conditions on Ob(**Loc**) (in such a situation one simply has that for any M

$$\mathcal{A}(M) \subseteq \mathcal{A}(\bigcup_{N\in\mathrm{Ob}(\mathbf{Loc})} N) \cong \varinjlim_{N\in\mathrm{Ob}(\mathbf{Loc})} \mathcal{A}(N) \tag{36}$$

because any locally covariant QFT obviously commutes with colimits –which in a posetal concrete category can be thought to be *joins*–). Let's denote $\mathcal{A}(M)$ as $\mathcal{A}^\circ$ and call it the *quasilocal* algebra associated to the AQFT $\mathcal{A}$.

In fact we have to remark that the present approach is slightly more general than the classical one, which prevents us from any size-concernment on the domain category of $\mathcal{A}$: in that framework we fix once and for all an object $M \in \mathbf{Loc}$ and consider the category $\mathbf{K}(M)$ of all open, relatively compact and causally convex subsets of M. This becomes a poset in the obvious way. Furthermore, any $U \in \mathbf{K}(M)$ can be seen as isometrically embedded in M in the obvious way, hence for any such U we have an isometric embedding $\iota_U : U \hookrightarrow M$: we can hence state the following

Theorem 4.1 [Araki-Haag-Kastler AQFTs]: *Let $\mathcal{A}$ be a locally covariant QFT and define a correspondence $\tilde{\mathcal{A}} \colon \mathbf{K}(M) \to \mathbf{Obs}$ sending $U \mapsto \tilde{\mathcal{A}}(U) = \mathcal{A}(U) \subseteq \mathcal{A}(M)$; then*

- *$U \subseteq V$ implies $\tilde{\mathcal{A}}(U) \subseteq \tilde{\mathcal{A}}(V)$ for any two $U, V \in \mathbf{K}(M)$;*
- *$[\tilde{\mathcal{A}}(U), \tilde{\mathcal{A}}(V)] = 0$ for any two causally separated $U, V \in \mathbf{K}(M)$.*

Then $\tilde{\mathcal{A}} \colon \mathbf{K}(M) \to \mathbf{Obs}$ can be seen to define a copresheaf *on $\mathbf{K}(M)$.The essential image of this functor is contained in the posetal category $C^*_{\subseteq}(M) \leq \mathbf{Obs}$, whose objects are the C^*-subalgebras of $\mathcal{A}(M)$ and whose morphisms are unital embeddings, so any AHK-QFT can be regarded as a $C^*_{\subseteq}(M)$-valued presheaf on $\mathbf{K}(M)$.*

In a minimalist approach, Araki-Haag-Kastler AQFTs can be explicitly defined only on a *basis* of M, exploiting the fact that (co,pre)sheaves on a space are uniquely determined from this assignment: we use **[Penrose]** and **[Comeau2]**, §**2.3** as references to describe such a basis of *open double cones* generating the *Alexandrov topology* on M.

First of all, recall that given a lorentzian manifold (M, g) a vector $v \in T_pM$ is said to be *causal* if it is null or timelike; the *causal cone* of a vector is the set of all causal vectors w such that $\langle v, w\rangle < 0$; this definition is easily extendable to curves.

Now suppose that M is time orientable (this boils down to the existence of a nonvanishing time-vector field on M), and fix an orientation on any tangent space T_pM; then a vector is said to be *future-* or *past-pointing* according to the sign of v with respect to the chosen orientation. We can define the following relations on M, which turn it into a partially ordered set:

- $p \ll q$ if there exists a future-pointing *timelike* curve connecting p to q.

- $p < q$ if there exists a future-pointing *causal* curve connecting p to q.
- $p \leq q$ explains itself.

The *chronological future* of a subset $A \subseteq M$ is defined to be

$$I^+(A) = \{q \in M \mid a \ll q \text{ for some } a \in A\} \tag{37}$$

and the *causal future* of A is

$$J^+(A) = \{q \in M \mid a \leq q \text{ for some } a \in A\} \tag{38}$$

Reversing the inequalities gives the definition of the *chronological* and *causal past* of $A \subseteq M$.

Remark 6 : *The chronological past and future $I^{\pm}(q)$ of a point are open in M for any $p \in M$.*

This entails that every set

$$\mathrm{I}(p,q) = I^+(p) \cap I^-(q) \tag{39}$$

is open in M; the collection of all $\mathrm{I}(p,q)$ with p,q running in M form a basis for a topology on M, which is called the *Alexandrov topology*. This topology coincides with the manifold topology on M precisely if the former is Hausdorff or M is *strongly causal*: this amounts to ask that any point admits an open neighbourhood W which is $\ll$-convex, meaning that it cointains an interval $[x,y]_{\ll} = \{z \in M \mid x \ll z \ll y\}$ iff it contains the endpoints.

The keypoint now is to notice that the following theorem holds:

Theorem 4.2 : *Let X a topological space and $\mathcal{F}$ a (pre)sheaf on X; then $\mathcal{F}$ is essentially defined by how it acts on a basis $\mathfrak{B}$ of the topology on X.* Essentially *means that there exists a unique $\rho\mathcal{F}$ extending a (pre)sheaf defined on $\mathfrak{B}$.*

(see **[Moerdijk]**, Theorem **2.1.3**: if $\mathfrak{B}$ is a basis for the topology on a space X, then there exists an equivalence of categories $\mathrm{Sh}(X) \cong \mathrm{Sh}(\mathfrak{B})$, the category of presheaves on $\mathfrak{B}$ which satisfy the sheaf axiom on every open set $U \in \mathfrak{B}$). If we denote as $\mathbf{W}(M)$ the set of double cones on our lorentzian manifold M, then an AHK-aqft can be defined as a sheaf on $\mathbf{W}(M)$, which is equivalent to define it on the whole category of open sets in the lorentzian manifold M.

4.1 An application: premonoidal crossed products.

Let M be a Von Neumann algebra, seen as a subalgebra of $B(H)$ for some Hilbert space H; suppose a discrete group G acts on M. The *crossed product* Von Neumann algebra $M \vee G = \widetilde{M}_G$ is defined via two embeddings

$$M \xrightarrow{\pi} M \vee G \xleftarrow{\lambda} G \tag{40}$$

such that the image of G consists of unitaries in $\widetilde{M}_G$, and the two images are related by the conjugation-equation

$$\pi(g.a) = \lambda(g)\pi(a)\lambda(g)^* \tag{41}$$

(this can be easily written in diagrammatical terms).

The construction of $M \curlyvee G$ is classical: start with the Hilbert space H where M is represented, and define a new Hilbert space

$$\widetilde{H}_G := \left\{\zeta\colon G \to H \mid \sum_{g\in G} \|\zeta(g)\| < \infty\right\}. \tag{42}$$

Now define embeddings into $B(\widetilde{H}_G)$ by $\pi(a)(\zeta)\colon g \mapsto (g^{-1}.a)\zeta(g)$ and $\lambda(g)(\zeta)\colon u \mapsto \zeta(g^{-1}.u)$; the crossed product $M \curlyvee G$ is now defined to be $(\pi(M) \cup \lambda(G))''$.

This is easily seen to be categorified by replacing any occurrence of $B(H)$ with its higher-categorical counterpart $\mathbf{Hilb}_{\otimes H}$, and any Von Neumann algebra with a Von Neumann category.

First of all we need to categorify the action of G on M: this can be done in various ways, but we follow the idea in **[Comeau]** and consider G (like any monoid, see Example **2.2**) as a one-object premonoidal category. Then a G-action on M can be easily seen as a functor $G \to M$, regarding also M as a category; hence we are led to consider the category $\mathrm{Funct}(G, \mathbf{C})$ for a Von Neumann category $\mathbf{C} \leq \mathbf{Hilb}_{\otimes H}$, made by $\mathbf{C}$-valued functors from G, and to define in the unique possible way the equations the map $G \times \hom(K, K')$ must satisfy for any $\hom(K, K')$.

One can easily notice that for $K = K' = \mathbb{I}$ the action restricts to a lower-categorical one on $\hom(\mathbb{I}, \mathbb{I})$, which is in a natural way a Von Neumann algebra (see Proposition **3.1** and Corollary **1**): this makes the categorification even more evident.

Notice that the Hilbert space $\widetilde{H}_G$ defined above is isomorphic to the tensor product $H \otimes \ell^2(G)$, where $\ell^2(G)$ is the space of functions $f\colon G \to \mathbb{C}$ such that $\sum_g |f(g)| < \infty$; then a basis of $\widetilde{H}_G$ is given by $\mathscr{B} = \{e_i \otimes \delta_g\}_{i\in I, g\in G}$, $\{e_i\}$ being a basis of H and $\{\delta_g\}$ is the basis of $\ell^2(G)$ made by Kronecker's deltas.

Viewing again $G = G[1]$ as a one-object category, we can define $\mathcal{L}\colon G[1] \to \mathbf{Hilb}_{\otimes \widetilde{H}_G} \leftarrow \mathbf{C}\colon \mathcal{P}$, in order to embed both $\mathbf{C}$ and G into $\mathbf{Hilb}_{\otimes \widetilde{H}_G}$. The strong premonoidal functor $\mathcal{L}$ amounts to a map $G \to B(\widetilde{H}_G)$ which can be defined on the basis $\mathscr{B}$ mimicking the classical definition:

$$\mathcal{L}(g)\colon e_i \otimes \delta_h \mapsto e_i \otimes \delta_{gh}. \tag{43}$$

The premonoidal functor $\mathcal{P}\colon C \to \mathbf{Hilb}_{\otimes \widetilde{H}_G}$ acts as the identity on objects and sends $f\colon X \otimes \widetilde{H}_G \to Y \otimes \widetilde{H}_G$ in $\mathbf{Hilb}$ to

$$\mathcal{P}(f)\colon x \otimes e_i \otimes \delta_g \mapsto \Big((g^{-1}.f)(x \otimes e_i)\Big) \otimes \delta_g. \tag{44}$$

Definition 4.4 (Premonoidal-categorical crossed product)**.** Let $\mathbf{C}$ a Von Neumann category, and G a discrete group regarded as a one-object premonoidal category. Let $\mathbf{A}(\mathbf{C}, G)$ be the union of the essential images of the functors $\mathcal{P}, \mathcal{L}$ defined above; then the *crossed product* of $\mathbf{C}$ and G is defined to be $\mathbf{C} \curlyvee G := \mathbf{A}(\mathbf{C}, G)''$.

4.2 A further application: categorification of AQFT.

4.2.1 The category of transportable endomorphisms...

Let's recall the classical framework in which *-monoidal categories arise in AQFT. First of all let $\mathcal{A}$ a fixed AHK-AQFT and $\pi_0 \colon \mathcal{A}^\circ \curvearrowright H_0$ a fixed representation[3] which we called the *vacuum* representation.

Now define the *complement algebra* of a double cone $U \in \mathbf{W}(M)$ to be the C*-subalgebra generated by the set $\bigcup_{V \perp U} \mathcal{A}(V)$, where $V \perp U$ means that V, U are spacelike-separated. Denote it as $\mathcal{A}^{\complement}(U)$. A *-morphism $\varrho \colon \mathcal{A}^\circ \to \mathcal{A}^\circ$ is said to be

- *localized* in $U \in \mathbf{W}(M)$ if it acts as the identity on $\mathcal{A}^{\complement}(U)$.
- *localizable* if it is localized on some $U \in \mathbf{W}(M)$.
- *transportable*, if for any $U \in \mathbf{W}(M)$ it is intertwined via a unitary element $u \in \mathcal{A}^\circ$ to a U-localized *-morphism ϱ_U:

$$\varrho(a) = u^{-1} \varrho_U(a) u. \tag{45}$$

Then we can define a category $\Delta = \Delta_{\mathcal{A}^\circ}$ whose objects are transportable *-endomorphisms of $\mathcal{A}^\circ$, and whose morphisms $\varrho \to \sigma$ are intertwiners between the two, namely $r \in \mathcal{A}^\circ$ such that $r\varrho(a) = \sigma(a)r$ for any $a \in \mathcal{A}^\circ$; composition of arrows is given by multiplication in $\mathcal{A}^\circ$.

The category Δ is easily seen to inherit a *-structure, and if some rather weak assumptions on the poset $\mathbf{W}(M)$ are satisfied it can also be turned into a category with biproducts. D'altra parte si puo' provare qualcosa di piu forte:

THEOREM 4.3 : *The category Δ is a* C**-category.*

Te proof is worked out in full detail in **[Comeau2]**, Theorem **8.2.12** where the explicit monoidal structure on Δ is defined and studied: the fundamental assumption to give Δ a monoidal structure is that the composition of localizable and transportable *-morphisms is again localizable and transportable.

Introduction of the category Δ can be motivated by the final remark contaned in Theorem **4.4**, which needs a little preparation.

Definition 4.5 (DHR representations)**.** A representation of an AQFT $\mathcal{A}$, $\eta \colon \mathcal{A} \to B(H)$ is called a DHR-representation if for each double cone $U \in \mathbf{W}(M)$ there exists a unitary map $T_U \colon H \to H_0$ (the base-space of the vacuum representation) such that

$$T_U \circ \pi(a) = \pi_0(a) \circ T_U \qquad \forall\, a \in \mathcal{A}(U^{\complement}) \tag{46}$$

where π is the (*algebraic*, not sheaf theoretical) representation of the quasilocal algebra $\widetilde{A}$.

We can form a category **DHR**-Rep having as objects all DHR representations of a given AQFT $\mathcal{A}$ where morphisms between two such representations are given by intertwiners. The main result (Theorem **8.4.3** in **[Comeau2]**) is

[3] Throughout this note a *representation* of an AHK-AQFT $\mathcal{A}$ consists of a natural transformation $\eta \colon \mathcal{A} \to B(H)$ (the second is the constant functor $U \mapsto B(H)(U) = B(H)$; a slight but straightforward effort can lead to a more general definition).

Theorem 4.4 : *The assignment* $\Delta(U) \ni \rho \mapsto \mathcal{J}(\rho) = \pi_0 \circ \rho$ *extends to a functor on the whole* Δ*, taking values in* **DHR**-*Rep.*

If the vacuum representation π_0 *s faithful and satisfies* Haag duality *then the functor* $\mathcal{J}$ *is an equivalence of categories* $\Delta \cong$ **DHR**-*Rep.*

By a simple "structure-transport" argument the category **DHR**-Rep can hence be endowed with a symmetric C*-structure.

4.2.2 ...and its categorification.

Everything that follows is new material proposed by **[Comeau2]** to categorify the definition of a aqft.

Definition 4.6. Let $(K, \preceq)$ be a directed poset; then a *local system of premonoidal* C*-*categories* consists of a functor $\mathcal{A}\colon K \to \mathrm{pC}^*$-**Cat** from K to the (2-)category of small premonoidal C*-categories and premonoidal C*-functors between them, such that for any $U \preceq V$ the functor $\mathcal{A}(U) \to \mathcal{A}(V)$ is faithful.

In view of the last request, it's easily seen that the correct domain for such a local system is the posetal 2-category pC^*-$\mathbf{Cat}_{\subseteq}$ of premonoidal C*-categories and premonoidal *faithful* C*-functors between them.

We can mimick the construction of the quasi local algebra $\mathcal{A}^\circ$ defining

$$\mathcal{A}^\circ := \Big(\bigsqcup_{U\in K} \mathcal{A}(U)\Big)/\!\sim \tag{47}$$

to be the category whose objects are the disjoint union of all $\mathcal{A}(U)$ quotiented out by the relation $(U, A \in \mathrm{Ob}_{\mathcal{A}(U)}) \sim (V, B \in \mathrm{Ob}_{\mathcal{A}(V)})$ if and only if there exists a $W \succcurlyeq U, V$ such that the objects $i_{UW}(A)$ and $i_{VW}(B)$ are equal in $\mathcal{A}(W)$. This relation is clearly an equivalence on the class of objects, and a similar relation defined for morphisms, where the elements are $(U, f \in \hom_{\mathcal{A}(U)}(A, A'))$ quotiented out by the relation which identifies $(U, f), (V, g)$ if they are eventually equal, turns out to be a congruence on te class of morphisms of $\mathcal{A}^\circ$.

The basic result which allows the development of such theory is the following:

Lemma 4.1 : *If* $\mathcal{A}$ *is a local system of premonoidal* C*-*categories, then* $\mathcal{A}^\circ$ *has a premonoidal structure.*

Proof. Refer to **[Comeau2]** for a complete proof: the tensor is defined to be the correspondence $(U, A \in \mathcal{A}(U)), (V, B \in \mathcal{A}(V)) \mapsto (W, i_{UW}(A) \otimes i_{VW}(B))$, where $W \succcurlyeq U, V$ (this is a well-posed definition since K is directed). ◻

The most natural prosecution of this process should be the presence of an isomorphism

$$\mathcal{A}^\circ \cong \varinjlim_{U\in K} \mathcal{A}(U). \tag{48}$$

This is not the case, since (see Example **10.1.8** in **[Comeau2]**) to obtain a C*-category one has to undergo a *completion* process. This point is precisely explained via an analogy in **[Comeau2]**:

> Suppose that $\mathcal{A}$ is a local system of C*-algebras, regarded as one-object C*-categories; Then we can costruct the category $\mathcal{A}^\circ$ which amounts to taking the union of all subalgebras of the quasilocal algebra. This turns out to be an algebra since K was directed (define operations "eventually" in the set-theoretic colimit), but in general it is not complete, hence it is not a C*-algebra.

The process of categorification allows to mantain a certain analogy with this counterexample, and suggests that $\mathcal{A}^\circ$ is not the right colimit due to completion problems.

However, $\mathcal{A}^\circ$ doesn't fall so far from the right colimit: the algebra $\mathcal{A}^\circ$ can be faithfully embedded in a premonoidal C*-category $\mathbf{U} = \mathbf{U}(\mathcal{A})$ via a surjective premonoidal C*-functor, and $\mathbf{U}$ turns out to be the object part of the 2-categorical colimit of the functor $\mathcal{A}$ (see **[Comeau2]**, Theorem **10.1.5**, Proposition **10.1.7** and Example **10.1.8**).

Te next step is to insert Einstein causality: let $(K, \le)$ be the poset of open double cones in Minkowski space, ordered by inclusion of subsets.

Definition 4.7 (Premonoidal C*-QFT)**.** A *premonoidal* C*-QFT consists of a local system of premonoidal C*-categories $\mathcal{A}\colon (K,\le) \to \mathrm{pC}^*\text{-}\mathbf{Cat}$ subject to the additional condition of Einstein's causality:

> Whenever $U \perp V$, then $\mathcal{A}(U)$ and $\mathcal{A}(V)$ *commute* in $\mathbf{U}(\mathcal{A})$, namely $\mathcal{A}(U)' \ge \mathcal{A}(V)$, $\mathcal{A}(V)' \ge \mathcal{A}(U)$.

The final task is to follow section **4.2.1**, which built the categories Δ and **DHR**-Rep and showed the link between the two (i.e. the functor $\mathcal{J}$), and mimick the same construction for a premonoidal C*-QFT.

Definition 4.8. Let $\mathcal{A}$ be a premonoidal C*-QFT endowed with a representation (π_0, H_0) (i.e. a functor $\pi_0\colon \mathrm{U}(\mathcal{A}) \to \mathbf{Hilb}_{\otimes H_0}$) called *vacuum representation*. A *premonoidal* DHR*-representation* consists of a representation (H,π) such that for any $U \in \mathbf{W}(M)$ there exists a family of unitary morphisms $\beta(U)_A\colon \pi_0(A)\otimes H_0 \to \pi(A)\otimes H$, for each $A \in \mathrm{Ob}(\mathbf{U}(\mathcal{A}))$ and such that for any $f\colon X \to Y$ in $\mathcal{A}(V)$, $V \perp U$, the diagram

$$\begin{tikzcd} \pi_0(X)\otimes H_0 \arrow[r, "\pi_0(f)"] \arrow[d, "\beta(U)_X"'] & \pi_0(Y)\otimes H_0 \arrow[d, "\beta(U)_Y"] \\ \pi(X)\otimes H \arrow[r, "\pi(f)"'] & \pi(Y)\otimes H \end{tikzcd} \tag{49}$$

commutes in the category **Hilb**.

With a careful definition of "morphism of representation" in this higher-categorical setting we can arrange premonoidal DHR-representations into a category, called again **DHR**-Rep.

The circle will be closed once we have built the higher-categorical counterpart of the category Δ of transportable morphisms.

Definition 4.9 (Localized C*-functor)**.** A premonoidal C*-functor $\mathcal{F}\colon \mathbf{U}(\mathcal{A}) \to \mathbf{U}(\mathcal{A})$ is termed *localized* at an open $U \in \mathbf{W}(M)$ in case that for each $V \perp U$ and object A in $\mathcal{A}(V)$, here is a unitary map $v_A\colon \mathcal{F}(A) \to A$ such that for any $f \in \hom_{\mathcal{A}(V)}(X, Y)$ the diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "v_X"'] & \mathcal{F}(Y) \arrow[d, "v_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{50}$$

commutes in $\mathbf{U}(\mathcal{A})$. A functor $\mathcal{F}$ is *localizable* if it is localized on *some* $U \in \mathbf{W}(M)$.

Definition 4.10. A functor $\mathcal{F}\colon \mathbf{U}(\mathcal{A}) \to \mathbf{U}(\mathcal{A})$, localized at $U \in \mathbf{W}(M)$ is said to be *transportable* if for any $V \in \mathbf{W}(M)$ there exists a premonoidal C*-functor $\mathcal{G}_V$ localized at V and a unitary premonoidal C*-transformation $\boldsymbol{\theta}\colon \mathcal{F} \Rightarrow \mathcal{G}$.

Let now $\mathbf{D}(U)$ the set of premonoidal C*-endofunctors of $\mathbf{U}(\mathcal{A})$ localized at $U \in \mathbf{W}(\mathcal{A})$. The union $\bigcup_{U\in\mathbf{W}(\mathcal{A})} \mathbf{D}(U)$ is the object part of a category, whose objects are transportable premonoidal C*-endofunctors and whose morphisms are *transformations* (no naturality) between such functors.

References

[Baez] J. C. Baez, J. Dolan, *Categorification*. http://arxiv.org/abs/math/9802029

[Brunetti] R. Brunetti, K. Fredenhagen, *Algebraic approach to Quantum Field Theory*, http://arxiv.org/abs/math-ph/0411072.

[Coecke] S. Abramsky, B. Coecke, *A categorical semantics of quantum protocols*. in Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science 2004, IEEE Comput. Soc. Press, (2004).

[Comeau] M. Comeau, *Von Neumann Categories*, http://arxiv.org/abs/1209.0124

[Comeau2] M. Comeau, *Premonoidal *-Categories and Algebraic Quantum Field Theory*, Carleton Institute of Mathematics and Statistics, Ottawa 2012.

[Halvorson] H. Halvorson, M. Müger, *Algebraic Quantum Field Theory*, http://arxiv.org/abs/math-ph/0602036.

[Leinster] T. Leinster, *Higher Operads, Higher Categories*, http://arxiv.org/abs/math/0305049

[Moerdijk] I. Moerdijk, S. Mac Lane *Sheaves in Geometry and Logic: a first introduction to Topos Theory*, Universitext Springer 1994.

[Power] J. Power and E. Robinson, *Premonoidal categories and notions of computation*, 1993. http://www.eecs.qmul.ac.uk/~edmundr/pubs/mscs97/premoncat.ps

[Penrose] R. Penrose, *Techniques of Differential Topology in Relativity*, CBMS-NSF Regional Conference Series in Applied Mathematics (1972).

[Warner] Garth Warner, C*-*algebras*. https://digital.lib.washington.edu/researchworks/bitstream/handle/1773/16302/C-star.pdf